

The Poetry of Numbers

Revealing beauty in maths



Everything is mathematical

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Preface

I once read somewhere that poetry is “indemonstrable truth”. According to the dictionary, poetry attempts to manifest beauty or aesthetic feeling through word, verse or prose. For mathematics, the dictionary establishes that it is a deductive science that studies the properties of abstract entities, such as numbers, geometric figures or symbols, and their relationships. Although it should do, this definition does not include an essential fact – the poetic emotion and the sense of beauty that guides the mathematician in deciding which properties of numbers or other abstract entities to study. Scholars of language, who are more closely linked to the arts than men and women of numbers, do not seem to have noticed the inextricable link between mathematics and beauty, which has been said to be the true inspiration for the great, and not so great, mathematical discoveries.

The importance of aesthetic values in mathematics, which positions it halfway between art and science, establishes an inevitable connection between this, the most abstract of disciplines, and human sensibilities. “It may be surprising to see emotional sensibility evoked *a propos* mathematical demonstrations,” wrote Henri Poincaré, “which, it would seem, can only be of interest to the intellect. This would be to forget the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance. This is a true aesthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility.”

All of this, the beauty of mathematics, which is as real as it is difficult to appreciate, and a few other, related issues are the subject of this book. Its purpose is to illustrate the pleasant aesthetics of mathematics, and all the emotional paraphernalia that surrounds it, with a handful of well-selected examples. Thus, it does not seek to weave an elaborate theoretical discourse, or amass reasons, arguments or proclamations in favour of the theses championed here. Excess theorising on the beauty of mathematics would be as absurd as explaining, and I am not the first person to say so, why Beethoven’s *Ninth Symphony* is beautiful.

This illustrative handful of examples will be adequately presented in their historical and emotional context; they will form a mosaic of rich and interesting stories, citing relevant episodes in humanity’s evolution over the last 25 centuries. In order to highlight the complexity of human nature, which these examples and their circumstances suggest, I have tried to lend the text a certain narrative tension, particularly in moments of emotional intensity. And, of course, I have not

avoided a leisurely look at other more orthodox arts such as painting, literature and architecture (in fact quite the opposite) in order to suggest similarities or to insinuate differences.

Ultimately, mathematics is demonstrated truths; therefore, its universe cannot be too far removed from that of poetry.

Chapter 1

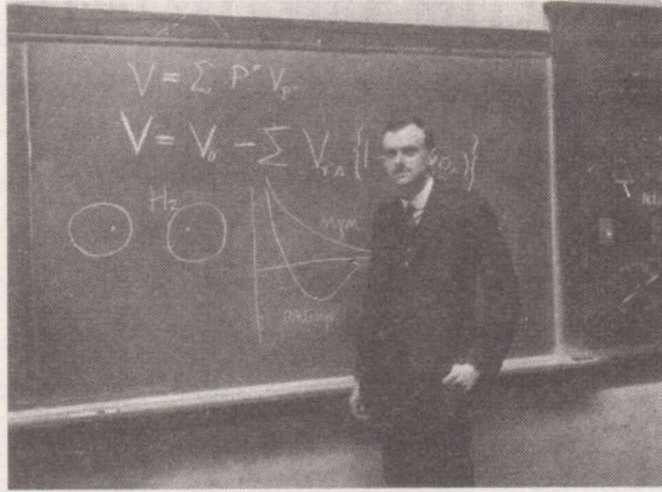
Where is the Beauty in Mathematics?

If we stop someone at random in the street and ask them about the beauty of mathematics they will undoubtedly look confused. However, in the collective imagination there is the impression that mathematics is impregnated with elegance and harmony, to the point of reflecting a certain unique beauty in its reasoning. Like so many other things in Western culture, the existence of this connection between beauty and mathematics probably began as a result of the judgement of those great commentators, the philosophers of Classical Greece. For Plato, the qualities of measurement and proportion, the essence of Greek mathematics, are a synonym for beauty. Meanwhile, in the words of Aristotle: “The chief forms of beauty are order and symmetry and precision, which the mathematical sciences demonstrate to a special degree.” Throughout history there has been a legion of scientists and thinkers who have praised the beauty of mathematics. In the 17th century, astronomer, astrologer and mathematician Johannes Kepler wrote that “geometry is the archetype of the beauty of the world”. More recently – in the 20th century – we find quotes such as this one from Bertrand Russell: “Mathematics, rightly viewed, possesses not only truth, but supreme beauty, a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature.” Another is from physicist and Nobel Prize winner Paul Dirac: “All physical laws should have mathematical beauty.”

And yet, despite all of this, nobody would be shocked by the surprise that would, in all probability, appear on the face of our anonymous passer-by as soon as we ask him or her about the beauty of mathematics. Most people would agree that it exists, but only a few have experienced its magic.

As an inevitable starting point, this book begins by proclaiming this special perception of our collective imagination: there is beauty in mathematics. However, I should immediately clarify that this beauty is difficult to appreciate. But all in good time; this first chapter is dedicated to explaining where the beauty that we are so sure exists in mathematics resides, while it will be in the next one that we discuss the

reasons why it is hard to appreciate. But first of all, it is fitting to start by explaining the terms that are to be discussed – beauty and mathematics.



Paul Dirac (1902–1984), a British physicist who made numerous contributions to quantum mechanics, is one of the many characters in science who have underlined the relationship between beauty and mathematics.

“Exalt the spirit”

Many thoughtful essays have been written about what beauty could be, and we have opinions that range from the unsettling to the downright charming. The verses of Rainer Maria Rilke (*Duino Elegies*) belong to the former: “For beauty is nothing but the beginning of terror which we are barely able to endure, and it amazes us so, because it serenely disdains to destroy us.” And this, from Stendhal, belongs to the latter: “Beauty is nothing other than the promise of happiness.” For the purposes of this book, it will not be necessary to get into any lengthy treatise in search of some sophisticated definition of beauty or aesthetic value. Looking up *beauty* in the dictionary will be sufficient. We will soon see the results of this adventure, because in this case, as is almost always true, turning to the dictionary has been an adventure.

First off we read: “Beauty, the quality or aggregate of qualities in a person or thing that gives pleasure to the senses or pleasurably exalts the mind or spirit.” I think this is a wonderful definition because it underlines the fact that the beauty of an object brings with it the capacity to affect us. In this, academics are in agreement with Voltaire who wrote in his philosophical dictionary: “It is not sufficient to see and to know the beauty of a piece of work. We must feel and be affected by it.” In the case of mathematics, it reflects the personal journey of most mathematicians I have

met throughout my scientific career and, of course, my own, very well. We attribute the attraction, curiosity and interest that we feel so strongly for mathematics (could it be love?) to the beauty that we find in it, to that thrill we feel when we dedicate our time to it, to the aesthetic pleasure we receive in return.

The hearty congratulations I proposed for the lexicographers for their dictionary definition of beauty falls on its face when I read the rest of the entry: “This property (beauty) exists in nature and in literary and artistic work.” To forget to include scientific work is an unforgivable error. How could I not use it to further illustrate my point!?

As far as the term *mathematics* goes, I will use it freely. It will, of course, include mathematical ideas, concepts and reasoning, and combinations of the three. Sometimes, I will use the expression ‘mathematical reasoning’ as a synonym for mathematics, in the broadest sense of the term, while on other occasions it will refer to more specific objects, be they theorems, definitions of concepts, demonstrations or even entire theories, and even reasoning that is more or less heuristic or has little logical foundation.

The Parthenon and Archimedes’ mathematics: edifying with ideas

In this book we accept it as given that there is beauty in mathematics. The fact that we understand that statement as an indemonstrable truth (with the poetic meaning that includes all indemonstrable truths), does not prevent us from discussing some of the unknowns that it throws up. Here I will try to respond to the most immediate question implied by the statement: there is beauty in mathematics, but where can we find it? Where does it reside?

An example will illustrate this better than theoretical discourse. And there is nothing better than establishing a parallel with the fine arts. As stated in the preface, this will be common in this book – referring to corresponding examples from art in order to illustrate and clarify points. (However, the reader should always take into account that, in some cases, they are controversial examples.)

So instead of wondering where beauty resides in mathematics we are going to ask ourselves where beauty resides in the Parthenon.

Construction of the Parthenon began in the Acropolis of Athens in around 447 BC, a few decades after the Persians ransacked the city and destroyed the old Acropolis. Its construction took almost a decade, although it still took a few more

years to complete its statues, which are mostly the work of the sculptor Phidias. After being a Greek temple, it has also served as a church, a mosque and, in the 17th century, the Turks converted it into a gunpowder depot. A Venetian bomb blew it up (1687), causing severe damage. At the beginning of the 19th century, British classicists took down many of the sculptural motifs that decorated the building's friezes and pediments, using their deterioration as an excuse, and they have been on display at the British Museum ever since, despite pleas from the Greeks to have them back.

The remains that are still in situ today, however, are enough for us to be able to answer the question: where does the beauty reside in the Parthenon?

Through detailed observation of the building we can confirm that its dimensions are in harmony. Its columns, of adequate proportions, have modifications to correct an optical illusion – they slope slightly inwards. Their diameters increase slightly at the centre, the end columns are slightly thicker; even the building's horizontal lines (the horizontal lines of entablatures and steps, for example) are curved so that they look straight, thus avoiding – in the words of George Santayana – the dryness and rigidity of a long straight line. (Perhaps this curvature, as the architect Oscar Tusquets Blanca noted, is less for aesthetics and more for practical reasons of the drainage of rainwater entering the peristyle.) And, in fact, there is also harmony in the decorative elements that adorned and fortified the building, even when placed in spaces as awkward as the truncated triangles that form the pediments of the main façades, and which we can now only enjoy by a stretch of the imagination – assisted by drawings, plans and the few remains kept in the British Museum.

So it is easy to conclude that the Parthenon's beauty resides in the harmony of the architectural elements that form it.

Based on this conclusion, we ask ourselves, what forms mathematical reasoning? It is clearly ideas, mathematical ideas. In other words, the beauty of mathematical reasoning must be sought in the harmonious combination of the various mathematical ideas in which they consist.

This conclusion, which we have reached here by a somewhat indirect route, was previously noted by G. Harold Hardy nearly three-quarters of a century ago in his essay *A Mathematician's Apology*. In this small book, which we will see again in more detail in Chapter 4, Hardy wrote: "A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. The mathematician's patterns, like the painter's or the poet's, must be beautiful; ideas, like colours or words, must fit together in a harmonious way. There is no permanent place for ugly mathematics."



Lateral view of the Parthenon. The damage caused in 1687, when the Turks' gunpowder store was blown up, can be seen.



Metope from the Parthenon which is currently exhibited at the British Museum in London. The work belongs to the friezes on the southern side, which were dedicated to the Centauromachy.

In the context of this conclusion, the time has come to give an example of the harmonious combination of mathematical ideas. I have chosen one which, for various reasons that I will explain later, is, as well as beautiful, very special: the calculation of the quadrature of a segment of a parabola, just as Archimedes did in *The Method of Mechanical Theorems*, one of his most important works. I will dedicate the rest of the chapter to this example as, taking into account the purpose of this book, it is essential to marinate it with historical spice whenever opportune. So before giving this example, I will give some details of the life and works of Archimedes – starting with his death, because Archimedes' death may be one of the most symbolically powerful images left behind by the ancient world.

The death of Archimedes and his skills as an engineer

The death of Archimedes is told in various chronicles from the ancient world and has been represented in a variety of formats throughout history, from a celebrated mosaic rescued in Pompeii, to the Pellegrino Tibaldi frescoes in the Monastery Library of El Escorial, or Delacroix's oil painting.

The year is 212 BC, the fifth since the conquest of Sagunto by Hannibal's troops triggered the Second Punic War. That year the Carthaginian general was still passing through Italy with his armies, although the war's focal point had shifted to Sicily and, more specifically, to Syracuse, an ally of Carthage, which was besieged by the Roman general Claudius Marcellus. Rome wanted to take complete control of the island and secure the cereal crops in the fields. After a failed attack and a long siege, the city fell into the hands of Marcellus, and his Roman troops prepared to rape and pillage. It was in the ensuing ransack that Archimedes was murdered. He was, at the time, more than 70 years old. It was chronicler Valerius Maximus who, two centuries later, was able to describe the events with the greatest drama:

“A soldier who had broken into Archimedes' house in quest of loot, with sword drawn over his head, asked him who he was. Too much absorbed in seeking a solution to a problem, Archimedes could not give his name but said, protecting the dust with his hands, ‘I beg you, don't disturb this,’ and was slaughtered as neglectful of the victor's command; his blood was confused with the labour of his science.”



This mosaic, excavated from Pompeii, shows a Roman soldier moments before decapitating Archimedes: the scholar's blood would be spilt over his notes.

The death of Archimedes should be considered as collateral damage of the invasion of Syracuse, because general Marcellus had given the order to spare the scholar's life. Plutarch, in the 1st century AD, recounted this interest in his *Life of Marcellus* – which includes as many as three versions of the death of Archimedes: “All agree that Marcellus was much grieved, that he turned away from his murderer as though he were an object of abhorrence to gods and men, and that he sought out his family and treated them well.”

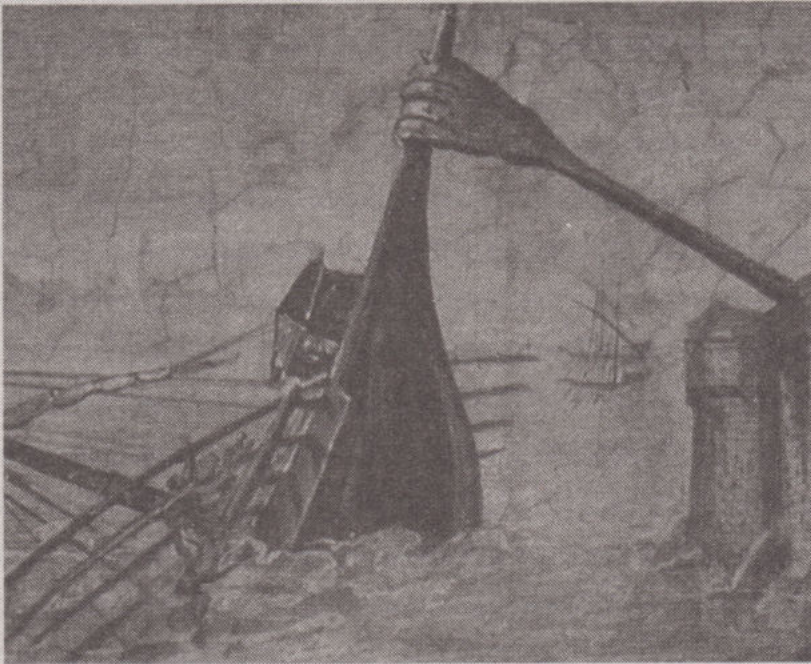
In all certainty, Marcellus' interest in Archimedes went beyond his fame as a mathematician – Archimedes was also a military engineer. Legend tells of the wonders of the superhuman power granted by his inventive capacity, almost to the point of elevating him to the status of a demigod: “All the secrets of the Universe were known to Archimedes,” wrote Silius Italicus in his epic poem *Punica* (1st century AD). “He knew when the dark rays of the rising Sun promised a storm, if the Earth was fixed or suspended on its axis, why the ocean that covers the globe remained chained to its surface, what caused the movement of the waves and the different phases of the Moon, what law was followed by the ocean in the ebb and flow of its tides. They said he had counted the sands of the Earth; he, who could launch a galley with the force of one single woman; he, who made piles of rocks climb against the slope of the terrain.”

The part about raising and lowering large weights, whether a galley or a rock, has always attracted the attention of the chroniclers. In Plutarch we also find an express mention of Archimedes' legendary ability with a lever: "Archimedes, in writing to King Hiero, who was a friend and close relation, had stated that, given the force, any given weight might be moved, and even boasted, we are told, relying on the

ARCHIMEDES AND HIS WAR MACHINES

The Classical chroniclers described truly horrific scenes caused by the destructive wrath of Archimedes' war inventions: "So when the Romans attacked," Plutarch recounted, "by sea and land at once, the Syracusans were at first terrified and silent, dreading that nothing could resist such an armament. But Archimedes opened fire on the Roman army and navy at the same time from his machines, throwing upon the land forces all manner of darts and great stones, with an incredible noise and violence, which no man could withstand; and their ranks were thrown into confusion, while some of the ships were suddenly seized by iron hooks, and by a counterbalancing weight were drawn up and then plunged to the bottom. Others they caught by iron-like hands or claws suspended from cranes, and first pulled them up by their bows till they stood upright upon their sterns, and then cast down into the water, or dashed them against the cliffs and rocks at the base of the walls, with terrible destruction to their crews. Often was seen the fearful sight of a ship lifted out of the sea into the air, swaying and balancing about, until the men were all thrown out or overwhelmed with stones from slings, when the empty vessel would

either be dashed against the fortifications, or dropped into the sea by the claws being let go."



Detail of a fresco by Giulio Parigi, painted at the end of the 16th century, which recreates one of the sophisticated war machines attributed to Archimedes.

strength of demonstration, that if there were another Earth, by going into it he could move this one using the other as a pivot for a lever.” The most extraordinary thing about the case of Archimedes is that he used his mastery of the lever not only for practical labour such as moving great weights; he also worked with the lever in the field of pure geometry, and with it carried out calculations as sophisticated as the area of a segment of a parabola which I have promised to show the reader. But this is not the time, because there are still other interesting details about Archimedes and the fame of his genius to be told.

Legends about Archimedes

Despite his ability as an engineer, legend has it that what Archimedes was really interested in was the matter of pure geometry. “And yet Archimedes,” wrote Plutarch, “possessed such a lofty spirit, so profound a soul, and such a wealth of scientific theory, that although his inventions had won for him a name and fame for superhuman sagacity, he would not consent to leave behind him any treatise on this subject, but regarding the work of an engineer and every art that ministers to the needs of life as ignoble and vulgar. He devoted his earnest efforts only to those studies the subtlety and charm of which are not affected by the claims of necessity.” And although it is true that no document written by Archimedes on military engineering has survived, this does not mean much. The condition of the work of Archimedes that has been discovered so far suggests that it is quite possible that he left documents on more practical issues, but they have not survived. Some of that applied character shows, for example, in his approximations of the number π (Archimedes demonstrates that it is a little less than $22/7$ and a little greater than $223/71$). It should be noted that Archimedes was an unusual mathematician in the ancient Greek world, and we have plenty of anecdotal evidence of his open-mindedness when it came to applied mathematics.

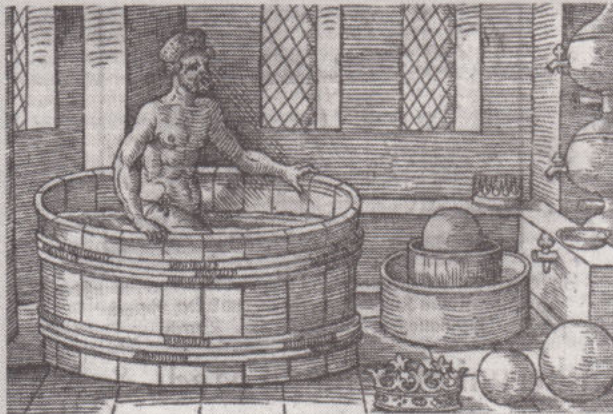
Those anecdotes have helped make the figure of Archimedes popular. Who has not heard “Give me somewhere to stand and I will move the Earth,” or “Eureka!”? Although they all have an element of truth, in many cases they are half the story. By way of example, let’s take a look at the famous legend which tells of how Archimedes discovered that Hiero’s crown was fake, using the different densities of gold and silver, with the universally known shout of “Eureka!” – subsequently becoming the scientists’ battle cry. The version provided by Vitruvius in the ninth of *The Ten Books on Architecture* states: “Hiero, after gaining royal power in Syracuse, resolved, to place

ARCHIMEDES' NUDITY

A naked Archimedes is not unique to the Eureka! story, and we find him as such, frisky and playful and his body smeared in oil by a private female assistant, in other scenes of an unequivocal hedonistic nature: "Archimedes, bewitched by some familiar and domestic siren," Plutarch stated in *Life of Marcellus*, "so as to forget to eat his food, and to neglect his person, and how, when dragged forcibly to the baths and perfumers, he would draw geometrical figures with the ashes on the hearth, and when his body was anointed would trace lines on it with his finger, for he was truly possessed by the Muses, by the joy he felt in his art."

This attitude of being "possessed by the Muses", akin to a scientist in complete concentration, has caused more than one complaint from the church, seeing in it a certain eroticism that linked scientific passion with sexual passion. The words of St. Augustine are infamous: "Besides that concupiscence of the flesh which consisteth in the delight of all senses and pleasures, wherein its slaves, who go far from Thee, waste and perish, the soul hath, through the same senses of the body, a certain vain and curious desire, veiled under the title of knowledge and learning." We can find an echo of St. Augustine's phrase in a phrase from Stephen Hawking, a present-day scientist: "There's nothing like the Eureka! moment of discovering something that no one knew

before. I won't compare it to sex, but it lasts longer."



A late 16th-century engraving dedicated to the famous "Eureka!" anecdote.

in a certain temple a golden crown that he had vowed to the immortal gods. He contracted for its making at a fixed price, and weighed out a precise amount of gold to the contractor. At the appointed time the latter delivered to the king's satisfaction an exquisitely finished piece of handiwork. But afterwards a charge was made that gold had been abstracted and an equivalent weight of silver had been added in the manufacture of the crown. Hiero, thinking it an outrage that he had been tricked, and yet not knowing how to detect the theft, requested Archimedes to consider the matter. The latter went to the baths while pondering the problem, and on getting into a tub observed that the more his body sank into it, the more water ran out over

the top. This observation led to the resolution of the case in question, and without a moment's delay, and transported with joy, he jumped out of the tub and rushed home naked crying "Eureka, Eureka!" – meaning "I get it, I get it!"

Like all accounts of this event, this one has very little scientific finesse, as Galileo rightly criticised it in his time, because what Archimedes did was undoubtedly much more refined than this report suggests. But, of course, if this had been included in Vitruvius' story it would have lost a lot of the drama, not least the image of a scientist running naked and shouting, which is much more striking than that of the same scientist studiously stooped over his desk. The latter is, of course, much more significant, but the former is much more attractive.

The account that tells how Archimedes burned a Roman fleet using mirrors falls decidedly on the side of legend and myth. The closest historical sources in Archimedes' time make no mention of the burning mirrors, although they do tell of his battle feats in quite exaggerated and grandiloquent terms. The authors who add this episode to the already bulging bag of Archimedean wartime inventions are much more recent. It is also doubtful that Archimedes had the technology necessary to manufacture such mirrors, although perhaps he understood the scientific foundations of such an invention, and this could be why its use as a weapon is attributed to him.

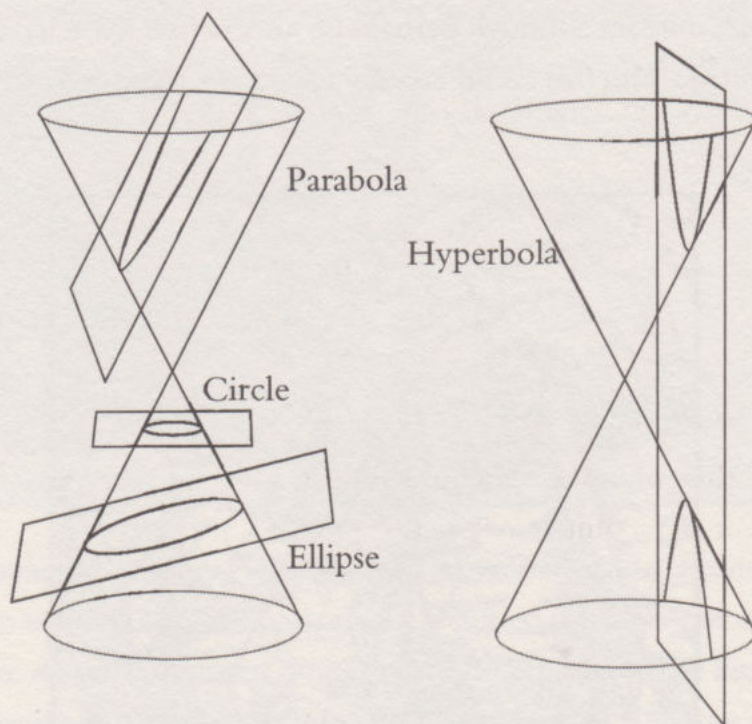


Engraving of the legendary use of the great mirrors to set alight the enemy fleet during the siege of Syracuse.

Put more accurately, Archimedes undoubtedly knew that a parabolic mirror would concentrate the Sun's rays onto a specific point (called the focal point). In case the reader is not familiar with it, the shape is more precisely known as a paraboloid of revolution which is formed by rotating a parabola on its axis. Archimedes demonstrated several, quite spectacular results on the parabola. One of them, its quadrature, is the example that I am going to use to demonstrate that the beauty of mathematics resides in the harmonious combination of mathematical ideas.

The quadrature of the parabola

The parabola, like the circle, the ellipse and the hyperbola, is a conical cross-section, or a curve obtained by intersecting a cone with a plane. Depending on the orientation of the plane, the cut through the cone will produce a circle, an ellipse, a hyperbola or a parabola. The latter is the curve obtained when the plane is orientated parallel to the cone's generatrix.

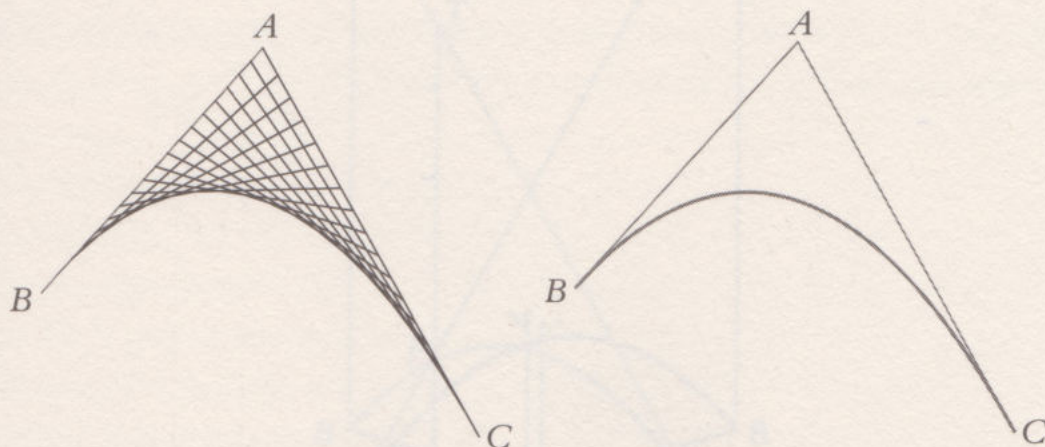


Complete family photo: the cone and its children.

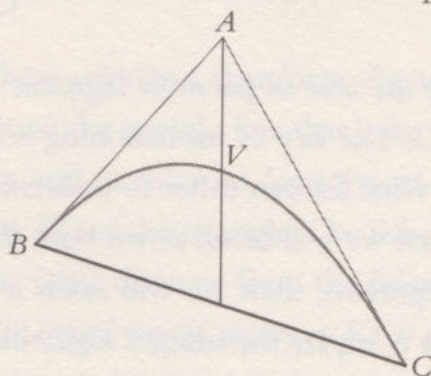
The Greeks attempted to square the area enclosed by each of these curves using only a ruler and a compass. They failed in the case of the circle and the ellipse – because the number π is required for their squaring – and also in that of

the hyperbola – for which they lacked logarithms – but they were successful in the case of the parabola, the quadrature of which was achieved by Archimedes in three different ways, all of which were equally ingenious. What I am going to explain below is taken from the work that we today know by the title of *The Method* (about which I will say a few things later, as its story is a fascinating one).

I will start by defining what a parabola is in more detail. Apart from being a conical section, the parabola can be defined in the following way. Let's imagine that we have an angle, the vertex of which is at A and is formed by two sides AB and AC . Let's call the ratio between their lengths r : $r = AB/AC$. Now select a point on side AC . It will be at a certain distance, which we are going to call d , from the vertex A . Join that point with the point on side AB which is at a distance $d \cdot r$ from B . If you do this with all the points on side AC , those segments 'enclose' a curve that is a section of the parabola. In the following figure on the left I have joined several points of segment AC with the corresponding points on AB , and on the right I have drawn the parabola and eliminated the segments that describe it.

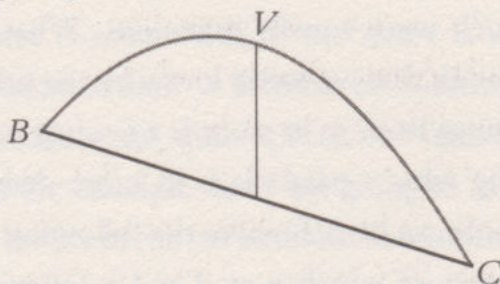


The axis of this parabola segment is the straight line that joins A with the mid-point of segment BC . Point V , where the axis cuts the parabola, is called the vertex.

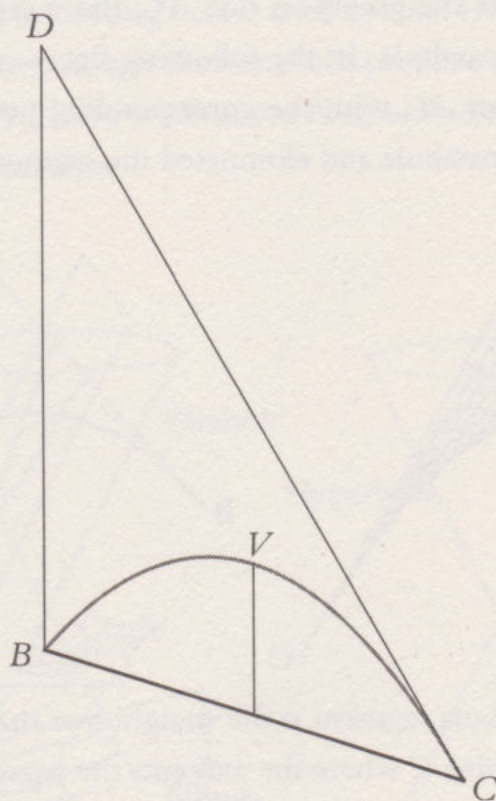


Parabola with its axis and vertex.

Let's consider then, parabola segment BVC , the vertex of which is at V , as shown in the following diagram.



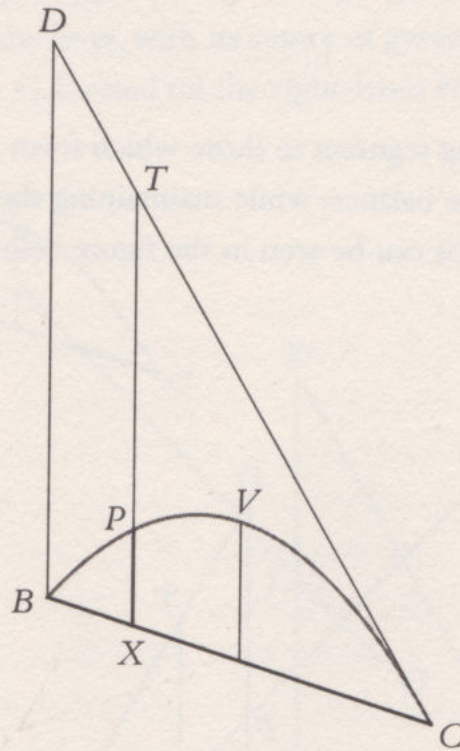
We are going to construct a triangle on the parabola segment with vertices D , B and C , as follows: side DB is parallel to the axis of the parabola segment and passes through B , while side DC is tangent to the parabola at point C .



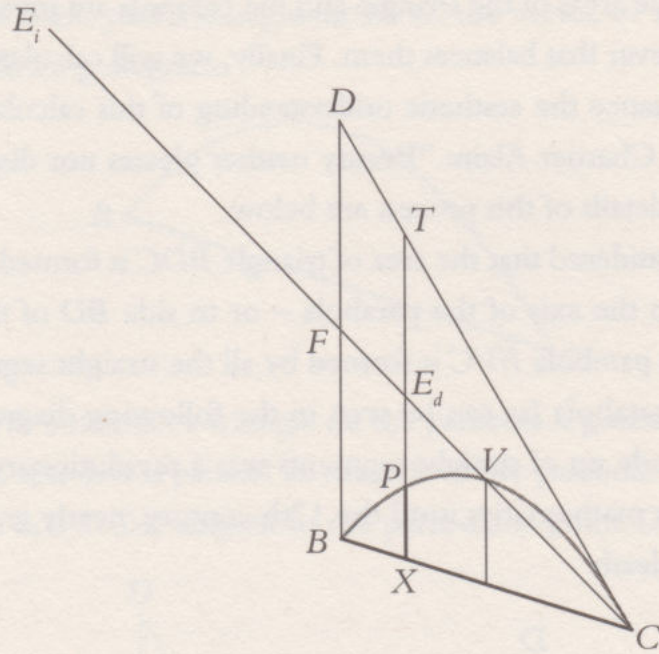
Archimedes showed that the area of parabola segment BVC is exactly a third of the area of the triangle BDC . The key to his reasoning is the mastery of the law of the lever. In order to make what follows easier to understand, here is a summary of what we are about to do. First we will break down both the area of the triangle and the parabola into straight segments; then we will insert a lever into the geometric figure which will allow us to compare the straight segments or separate parts that we have created from both figures. We will then rebuild the triangle and the parabola, each of which will be balanced on either end of the lever. The law of the lever tells

us that the respective areas of the triangle and the parabola are inversely proportional to the arms of the lever that balances them. Finally, we will calculate the value of that proportion. To enhance the aesthetic understanding of this calculation think of this phrase from Emile Chartier Alain: “Beauty neither pleases nor displeases us, instead it detains us.” The details of this process are below.

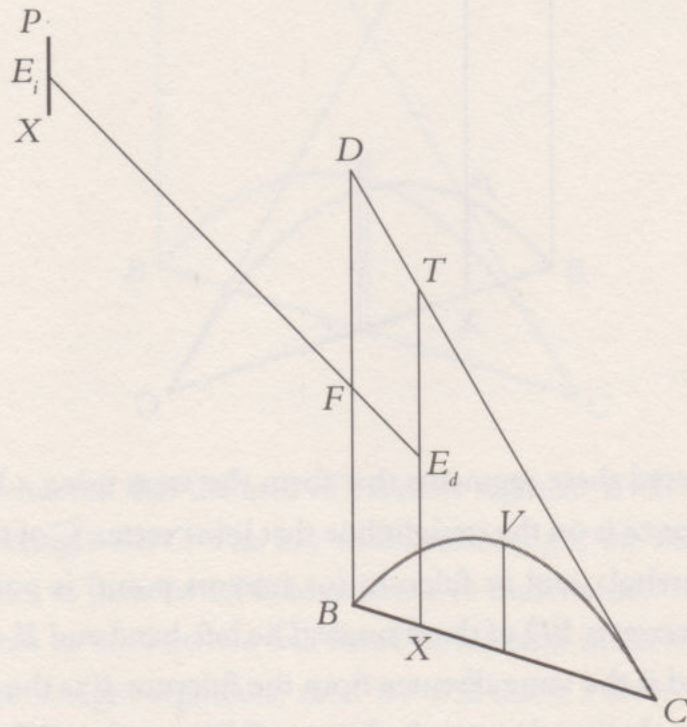
Archimedes considered that the area of triangle BDC is formed by all the straight lines XT parallel to the axis of the parabola – or to side BD of the triangle – and that the area of the parabola BVC is formed by all the straight segments XP parallel to the axis of the parabola (as can be seen in the following diagram). Considering an area as being made up of straight segments was a revolutionary idea that would not appear again in mathematics until the 17th century, nearly two thousand years after Archimedes’ death.



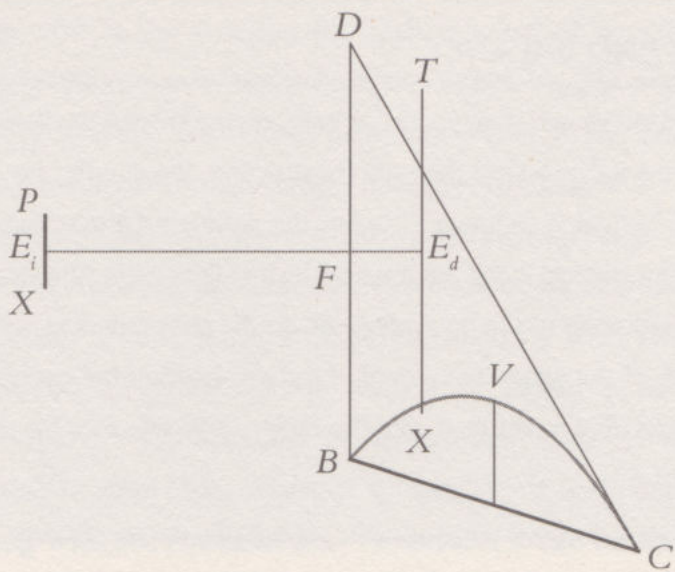
He then compared these segments that form the areas using a lever or balance. The arm of the balance is on the straight line that joins vertex C of the triangle with vertex V of the parabola, and its fulcrum (or support point) is point F where the arm of the lever intersects BD of the triangle. The left-hand end E_l of the balance is always the same and is the same distance from the fulcrum F as the fulcrum is from vertex C of the triangle (in other words, distance $E_l F$ is equal to FC). The right-hand end of the balance E_r varies, and it is the intersection of the arm of the lever with the corresponding segment to those that form the triangle.



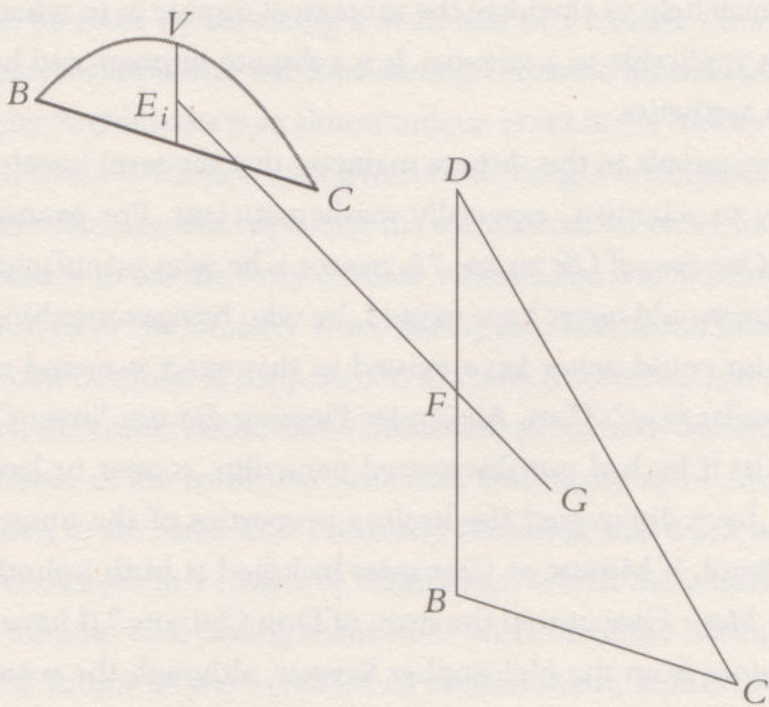
So, if the corresponding segment to those which form the parabola is moved to the left-hand end E_i of the balance, while maintaining the segment of the triangle at the right-hand end E_r (as can be seen in the figure below),



and the lever is left to oscillate, it will balance itself.



Consequently, moving the parabola segment by segment, Archimedes managed to balance the parabola on the lever, with its centre of gravity on E_p , with the triangle, with its centre of gravity G , located on the right-hand end of the lever.



According to the law of the lever, this means that the areas of the parabola and triangle are inversely proportionate to the lengths of the arms of the balance they are resting on. So, this proportion is a third (for further details, see the text-box on the next page). Therefore the area of the segment of the parabola BVC will also be a third of the area of the triangle BDC .

PROPORTION AND BALANCE

Let's take a detailed look at why the proportion between the arms of the lever where the triangle and the parabola are balanced is a third. By construction, the length of the left-hand branch of the balance EF is equal to segment FC , while the right-hand arm is segment FG . Now, the centre of gravity of a triangle is the point where its mid-lines meet (the straight lines that join a vertex with the mid-point of the opposite side); on the corresponding mid-line the centre of gravity is two thirds of the vertex and a third of the mid-point of the opposing side. As FC is a mid-line of a triangle (it joins vertex C with the midpoint of side BD), the distance FG will be, therefore, a third of FC .

Mathematics: creation or discovery?

The way in which Archimedes calculated the quadrature of the parabola, which you have just seen, may help to elucidate the subtextual dispute as to whether or not the term 'creator' is applicable to a scientist. It is a dispute impregnated by issues linked to thoughts on aesthetics.

Perhaps most people in this dispute maintain that the term 'creators' should not generally apply to scientists, especially mathematicians. For example, Fernando Savater in *The Questions of Life* wrote: "A creator is he who manufactures something that without him would never have existed, he who brings something to the world that without him could never have existed in that exact way and not in another more or less similar way." Thus, Alexander Fleming did not 'invent' penicillin but he discovered it: if he had not discovered penicillin, sooner or later some other scholar would have discovered the healing properties of the miraculous fungus. On the other hand, if Mozart or Cervantes had died at birth nobody would have composed *The Magic Flute* or told the story of Don Quixote." (I have illustrated my point with a quote from the philosopher Savater, although the words of scientists such as François Jacob – a winner of the Nobel Prize for Medicine – in *Of Flies, Mice and Men*, for example, could also have been used.)

However, there are two planes in scientific fact. One is that of discoveries – be they theorems, universal laws, a galaxy or a chemical element – and the other is the way in which the proof or discovery is made. Because of these discoveries, it clearly seems more appropriate to call a scientist a 'discoverer'. But there are times

– perhaps not many, but a few – when the qualification of ‘creator’ can be fittingly used for the way in which a scientist proves or makes his discovery.

Thus, we can say that Archimedes is not the creator of the relationship existing between the areas of the parabola and the triangle, because if Archimedes had not discovered it, someone else may have done sooner or later. But Archimedes did not only establish the relationship between these areas, he also did it in a particular way; and it is precisely this specific way of establishing the relationship between these areas by balancing them on a lever that can be called an act of creation. This means of discovery carries Archimedes’ unmistakable stamp. In the same way that we cannot imagine *The Hay Wain* without Constable, it is difficult to imagine this geometric reasoning without Archimedes. Therefore, it can be said that Archimedes discovered the formula for squaring the parabola, but his procedure of breaking it down into segments and *weighing* them, with such aesthetic value, is his own creation in the complete sense used before by Savater: “Without him it could never have existed in that exact way, nor in another more or less similar way.”

And when I say that if Archimedes had died at birth nobody would have squared the area of the parabola by balancing it with that of a triangle on a lever, I am not expressing a personal opinion, but formulating verifiable historic fact. In this sense, this reasoning by Archimedes is an almost unique event in the history of science. And its singularity is owed to the fascinating historical changes undergone by *The Method*, the work where Archimedes explained his calculation. In order to understand my point it is necessary to tell the story of these vicissitudes, which are none other than those that occurred to Archimedes’ work during its transmission down the centuries to the present day. Because, as happened to so many intellectual and artistic works in the classical era, the conservation and transmission of Archimedes’ work was hanging on a thread. Some of his work was even lost; and nearly all of Archimedes’ work could have fallen to the same fate. For many centuries, this work was only known to have been conserved in a couple of manuscripts which the whirlwinds of history shook like an autumn leaf, tossing them from one side of the Mediterranean to the other while the drums of war rumbled all around them, armies were looting and entire cities were burnt down.

Archimedes’ *The Method* and written sources

The oldest manuscripts containing work by Archimedes of which we have historical evidence were composed in Constantinople in the 10th century, or at a stretch,

in the 9th. Previously there must have been others that fill the gaps going back to the original documents written by Archimedes in the 3rd century BC, but all those manuscripts have since been lost.

Archimedes undoubtedly wrote all or most of his work in the (scientific) solitude of this native Syracuse. He was born there in 287 BC, although when he was a young man he studied in Alexandria, the Hellenistic centre of mathematical and scientific knowledge (it would continue to be so until the 5th century AD). From there he returned to Syracuse, where he lived for most of his life. Archimedes' scientific output that has been preserved consists of research papers, in modern terms, which were created throughout his life and were sent from Syracuse to Alexandria, or even to Samos, where Conon, one of his most admired friends lived. Among those monographs is the one that interests us here, *The Method of Mechanical Theorems*. It is a long letter that Archimedes addresses to Eratosthenes, when he was the chief librarian of the Ancient Library of Alexandria, in which he explains the method he used for his discoveries.

It is very possible that each piece of Archimedes' work followed the same route to Alexandria, with a complete work only being formed after his death. The mathematical magnitude of Archimedes' work is considerable, and far superior to that of Euclid's *Elements*. The basic level of most of the *Elements* leads us to suppose that it was widely proliferated, but that did not happen with the more sophisticated Archimedean work, which was only suitable for experts. It is reasonable to think that there were very few copies, possibly kept in the great Library of Alexandria, or in the daughter library in the Serapeum. This dispersal of Archimedean work was undoubtedly responsible for the loss of some of it and the deterioration suffered by other pieces. This deterioration is already recognisable half a century after his death, where there is evidence that authors could not find some of his theorems. From other quotes, however, we know that in the 3rd and 4th centuries AD work by Archimedes was conserved which has since disappeared, it may have been lost in the devastation of the Serapeum in the year 391 AD.

In the first third of the 6th century there was an attempt to systematically gather, order and comment on Archimedes' work. We cannot be sure if this was the first such attempt, but there isn't any evidence of a previous one. We now go from Alexandria to Constantinople, just when the Eastern Roman Empire was transforming into Byzantium, during the transition from Emperor Justin to Justinian. The former was a rough, illiterate soldier while the latter was cultured

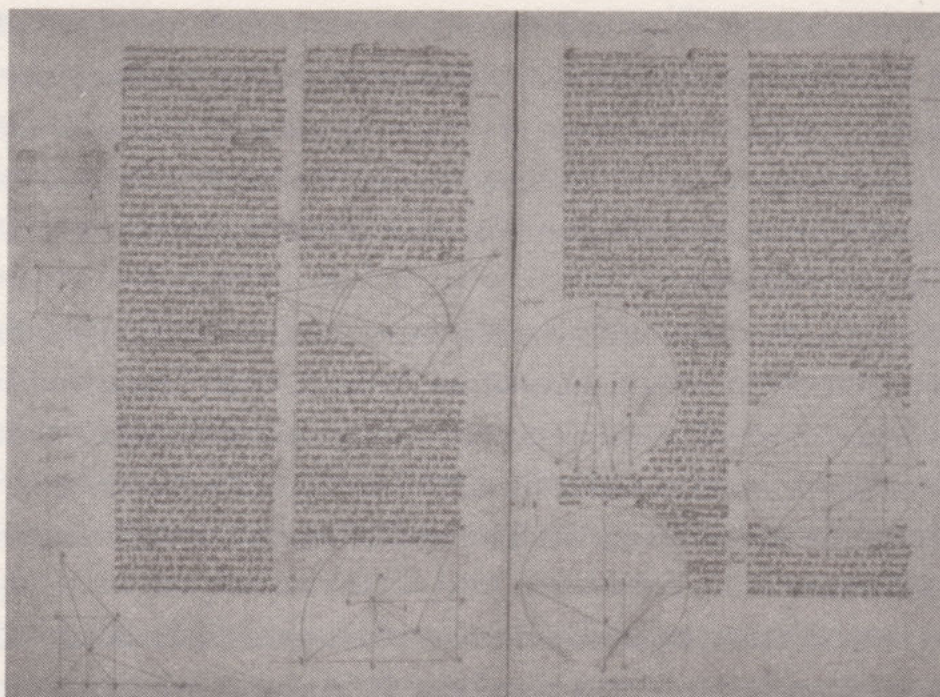
and well versed in theology and law. At the time there was renewed interest in great classical mathematics and, although it did not produce any relevant mathematics, it did have the virtue of preserving some fundamental work for the future – such as the work of Archimedes. In 529, Justinian ordered the closure of the Platonic Academy and other scientific and philosophical centres, accusing them of pagan teachings.

Three years later, the emperor decided to construct the Santa Sophia basilica. The architects of the new basilica were Anthemius of Tralles and Isidore of Miletus, who promoted conservation of the Greek scientific legacy, ordering that as much scientific work as was still available be copied and annotated. Eutocius compiled all the Archimedean work he could find, and wrote commentaries on three of them.

Two centuries later, Byzantium entered a new period of cultural, military and religious splendour. It was in that period that the three Greek manuscripts that have allowed the population of the last thousand years to discover Archimedes' work were put together. These three manuscripts appear to originate from the same city, Constantinople, and to have been composed between the 9th and 10th centuries. Of the three, only one has survived until today. But that one, perhaps as it was lost for a long time, has left little impact, to the point that it is the other two manuscripts (and if pushed, I would say just one of them) that are responsible for the enormous influence of Archimedes' work on European mathematics in the 17th century. At that time Archimedes was the mathematician most quoted by researchers (although his work was now nearly two thousand years old). In order to identify those three manuscripts, I will call them codices A, B and C. The A and B codices, either together or separately, were both moved from Constantinople to Sicily, the native land of Archimedes, in the 12th century.

Codex B, the most elusive of the three manuscripts, was probably a collection of work on mechanics and optics by various authors. It is now barely a ghost that disappeared at the beginning of the 14th century and of which we know little more than the fact that it was used in the 13th century to create a Latin translation of some of Archimedes' work.

Codex A led a turbulent life until its disappearance in the middle of the 16th century; but fortunately it left a cohort of descendants, copied between the mid-15th and the mid-16th centuries, which are still conserved. (The best four copies are kept at the Marciana Library in Venice, the Laurentiana in Florence and there are two in the national library of France.)



Double page from to Willem van Moerbeke's Latin translation of Archimedes' work.

The first print edition of the work of Archimedes was prepared, in Greek and Latin, from codex A and its descendants, and from the Latin version of codex B. It was published in Basel in 1544 and finally a large part of Archimedes' work became available to the Renaissance mathematicians.

But *The Method* did not form part of that work, as it did not belong to codex A or B. Codex C is the only one whose whereabouts we now know. It was discovered in 1906 by the scholar Johan Ludvig Heiberg, a Greek professor at the University of Copenhagen. In reality, codex C is a palimpsest, that is, an ancient manuscript that holds traces of a previously erased scripture. In this case, the mathematical content had been hidden when an euchology (a prayer book which contains the services of the Sabbath and the year's main Christian festivals) was written on top of it.

Codex C

Codex C has a fascinating history. It was possibly the last of the three Archimedean Byzantine codices to be composed. It is also the one that has had less influence throughout history, due to the fact that it had remained hidden until its discovery little more than a century ago, in 1906.

Judging by the calligraphic characteristics, the manuscript could have been composed around the year 975. Two and a half centuries later, someone decided that

there were other, more important things to write than what was contained in the manuscript, and so that person began to scrape away all the content in order to reuse the parchment. Pages from four other books were added to those of Archimedes; the pages of the parchment were inevitably mixed up, cut and rebound, the consequence of which was that the new text was written perpendicular to the ancient remains. In short, a devoted copyist converted Archimedes' manuscript into a palimpsest and wrote Christian prayers on the finest and most complex mathematical reasoning produced by the Greek world. Using ultraviolet light, a colophon has been read which confirms that the palimpsest was finished on 13 April 1229.

Archimedes' work remained buried under orthodox prayers, but time did its work and, little by little, curiosity led scholars towards the manuscript's earlier content. Thus, in the middle of the 19th century the erudite German Constantine Tischendorf, after visiting Constantinople, reported a palimpsest with mathematical content. And so the palimpsest began to reveal its secret. Tischendorf did not hesitate to pull a page of the manuscript out (not knowing that it contain several of Archimedes' theorems), a page that was sold to the University of Cambridge in 1876, where it remains today.

The next person to take notice of the manuscript was Greek palaeographer Papadopoulos Kerameus, who included it in a catalogue of manuscripts that he published in 1899. From the Archimedes palimpsest, Papadopoulos Kerameus managed to read a fair number of lines of the hidden text which he transcribed into his catalogue. According to Papadopoulos, the manuscript contained a notation from the 16th century (which is now lost) saying that the book belonged to the St. Sabbas Monastery in Palestine. Nothing is known of how or why the palimpsest came to be in a monastery-fortress hidden in the mountains to the south of Bethlehem. And so the palimpsest spent an unknown period of time in Palestine, although it was in Constantinople by the time that Tischendorf went there in 1840 and tore a page out.

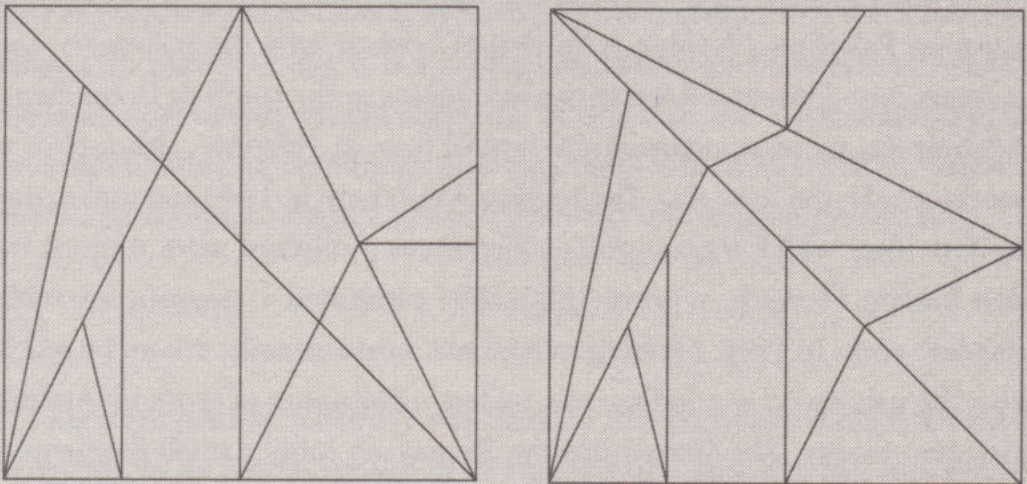
The few lines which Papadopoulos Kerameus published were of great interest to Johan Ludvig Heiberg, who in 1880-1881 published a magnificent edition of Archimedes' work. In 1906, Heiberg moved to Constantinople, where he recognised that what the palimpsest was hiding was, no less, a collection of work by Archimedes, of which two pieces, *The Method* and the *Stomachion* (only a small fragment of this was conserved), had not appeared in any of the manuscripts known until then. Another, *On Floating Bodies*, was only known through the mediaeval Latin translation of codex B. The finding of the codex C is undoubtedly the most significant event in recent centuries for understanding classic science. With the photos he took of

the manuscript, Heiberg prepared a new edition of work by Archimedes, naturally including *The Method*, which was published between 1910 and 1915. The integrity and depth of Heiberg's study is surprising, taking into account the lack of technology at his disposal, and the difficulty of reading the scraped text in Archimedes' work.

The history of Archimedes' manuscripts tells us that *The Method* was lost from mathematics almost since its composition until Heiberg made it public early in the 20th century. And with it came the calculation of the area of the segment of a parabola, as explained in the previous section. So, during that long two and a bit millennia we do not know of any mathematician who has calculated the area of the

STOMACH ACHE

Stomachion is a Greek word that means 'stomach ache', but it is also the name of a work by Archimedes (of which we only have an extract in Arabic and two very deteriorated pages of a palimpsest), and of a game, a type of geometric puzzle. The idea is to form squares (or other shapes) with the 14 predefined pieces into which a square has been divided. The difficulty of solving it could produce a headache or even a stomach ache, which is probably where the name of the game and the title of Archimedes' work comes from. But after studying codex C, it is now thought that Archimedes' *Stomachion* could have been a treatise on combinatorics. The discovery, if it is true, as it is still not confirmed or established (given the scarce and delicate state of the text that has been conserved), would be a genuine surprise, as combinatorics are far removed from Greek geometric operations and, in particular, those of Archimedes.

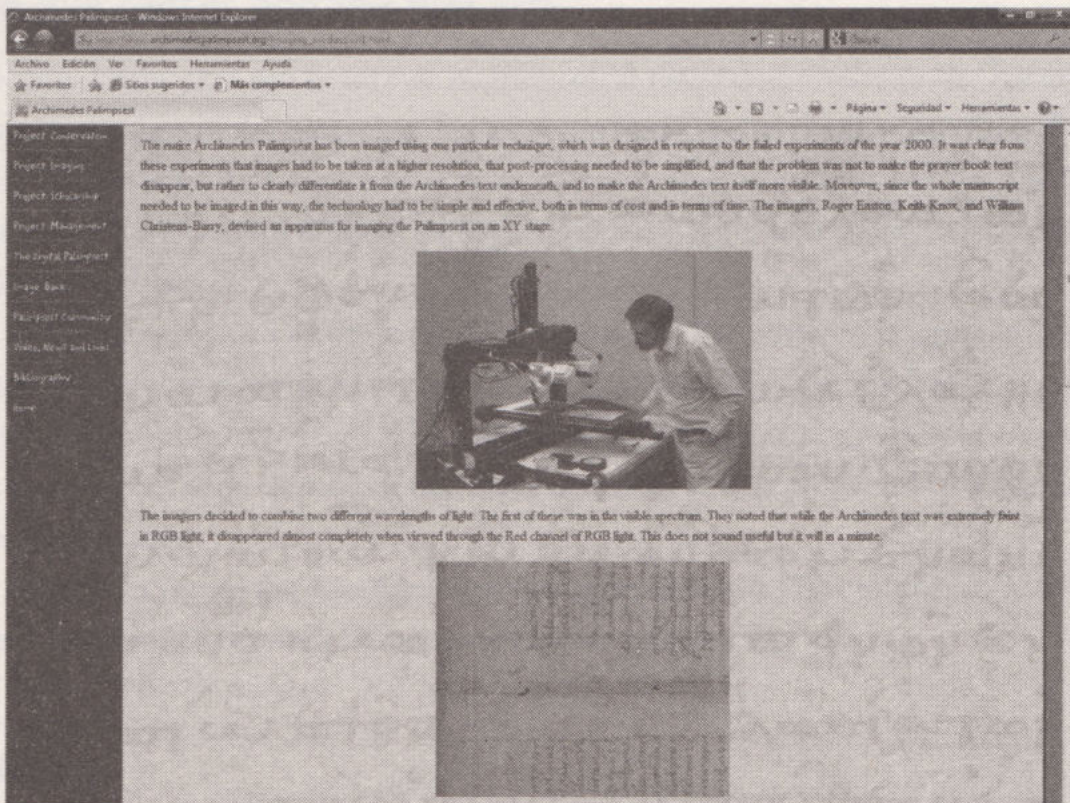


Left, the initial position of the pieces in the Stomachion. Right, one of the 17,152 different ways of rearranging the pieces to re-form the initial square.

parabola by balancing it with a triangle on a lever. This shows that if Archimedes had died at birth, this way of calculating the area of the parabola would never have existed in exactly this way and not in another more or less similar way, because as far as we know, nobody every repeated Archimedes' reasoning. This very unusual way of doing the calculation, so full of harmony and aesthetic value, therefore, has every right to be called a creative act.

The latest adventures of the Archimedes palimpsest

It would be unforgivable to end this section without reporting the latest adventures of codex C. When Heiberg published its content, the palimpsest was almost certainly stolen and disappeared again. It was lost for almost the entire 20th century until it reappeared on 28 October 1998, in New York, in an auction at Christie's. It was purchased, for over \$2 million by an anonymous American collector. A few months later, the collector left it at Walters Art Museum in Baltimore for its custody, conservation and study.



The website The Archimedes Palimpsest Project contains detailed information on the process that has allowed the Archimedes palimpsest to be recovered.

Since then intense work has been carried out on the conservation and examination of the palimpsest by scholars and experts in classical sciences, restorers and image treatment and processing technicians, using state-of-the-art technology. So, in this odyssey through the 20th century, the palimpsest has suffered more damage than during the rest of its history. Some pages have disappeared, many others have been seriously attacked and damaged by a mould that makes reading very difficult with the naked eye (this damage is noticeable when compared to the photos of the manuscript taken by Heiberg) and, finally, someone, presumably thinking that the price of the manuscript would be boosted, added four miniature paintings of the evangelists – damaging the pages on which they were painted.

Chapter 2

Why is it Difficult to Appreciate the Beauty of Mathematics?

As stated at the start of the previous chapter, nobody would be surprised that the hypothetical 'man on the street' would be confused about explaining or even confirming the existence of aesthetics in mathematics. Because we are assuming that this aesthetic value exists, this probable confusion can only mean one thing – the beauty of mathematics is difficult to appreciate. This leads us to the inevitable question, which gives this chapter its title.

The five senses and the fine arts

We now know that the beauty of mathematical reasoning resides in the harmonious conjunction of the ideas that it is made up from, just as the beauty of a building lies in the harmony of the architectural elements that form it. However, for most people it is much more difficult to appreciate the beauty of a theorem than that of a Gothic cathedral.

Why is this so? As I understand it, the answer to the question is a physio-logical issue. The problem we humans have in appreciating the aesthetic value of mathematical reasoning is the lack of a suitable sense for discerning, without thinking, the structure of ideas that comprise that reasoning, as it is presented in a harmonious combination.

Before discussing this statement I am going to illustrate a few examples that show how close the relationship between our senses and the fine arts is.

Painting

I will begin with painting. According to what was discussed above, we can state that the beauty of a painting lies in the harmonious combination of its elements, such as the shape, the colours, the composition, the space and the light, even the texture.

For utilitarian reasons I will consider a purely formalist vision of painting in this example, leaving aside the other aesthetic values (ethics, morals, etc.) or functions (social, religious, etc.) that it may encompass, to which I will refer later.

So, in each case, the corresponding pictorial element is captured directly by means of the sense of sight, as are the complete relationships between them.

Let's take a look at the cave painting in the illustration below. It is made up of a few simple coloured marks on the wall of a cave. Sight allows us to recognise that some of the painted shapes are animals, others are people, and that we are looking at a hunting scene. In a quick glance we have captured the whole structure of shapes in the painting; now the mind can make a decision on the harmony of its composition.



Cave painting from the Tassili Plateau, in the south-east of Algeria, a region that has been declared a World Heritage Site by UNESCO due to the wealth of its archaeological sites.

Likewise a glance at Van Eyck's painting titled *The Arnolfini Wedding* is enough to understand its colours; the brain automatically has the chromatic information available in order to make a decision on the beauty of the painting.

Again, our sight automatically perceives the composition of *School of Athens*, the fresco by Rafael in the Vatican Museums. The distribution of characters in symmetrical groups (among which we can recognise Pythagoras, Euclid, Ptolemy,



*The Arnolfini Wedding, oil painting by Jan van Eyck, 1434,
now on exhibit in London's National Gallery.*

Plato and Aristotle), their positioning within the succession of domes which houses the scene, the depth created by the perspective. All the information is automatically and quickly taken to the brain by our sense of vision where the harmonious combination of the elements is perceived.

Nothing escapes sight, not even the space or light created by Velázquez in *Las Meninas* which fill the entire scene in the painting; even the texture of the brush-



School of Athens, fresco painted by Raphael between 1510 and 1511 in the Vatican.



Left, Las Meninas, oil painting painted by Velázquez in 1656, today in the Museo del Prado, Madrid. Right, detail from The Sower (1888), one of several oil paintings with the same title by Vincent Van Gogh, which currently belongs to a private collection.

strokes in Van Gogh's *The Sower*, as if we were touching the painting, is captured by sight – almost superceding touch – and transmitted to the brain.

Music

Similar considerations can be made for music in relation to the sense of hearing. In this case, the temporal ordering of the musical chords, their kinetic character, etc. is considered. In the words of Monroe C. Beardsley: "As music is a temporal art it flows with time; it churns, swells, undulates, becomes dark and violent, it rises, falters, moves continuously." This temporal arrangement of music, which is irrelevant in painting, is also essential in mathematics. Put simply, a theorem, just like a symphony, starts, flows and ends, and the order in which that happens is fundamental to it.

The sequential nature of music has significant implications when it comes to enjoyment: as the melody transpires, we need a certain capacity for acoustic memory in order to appreciate its aesthetics correctly. And the fact is that in human beings this is not particularly well developed, if we compare it with, for example, our visual memory. I once heard someone say that, in this sense, a human listening to a Brahms quartet is quite similar to a fish watching Hitchcock's *Psycho*. Our immediate acoustic memory is not particularly equipped for retaining complex chains of



The Cellist, photograph taken by the photography pioneer Antonio Giulio Bragaglia in 1913.

sound, and even less so for recognising them several minutes later hidden behind a fairly subtle change in rhythm. It is certainly what would happen to a fish in front of a screen. After a few minutes, maybe seconds, having seen a scene it will have forgotten it already, being incapable of recognising a character who has not been on screen for a little while. I am afraid that this poor ability of the human being to retain complex melodies is also present in terms of the abstract subtleties of good mathematical reasoning. The consequence is that our limited ability to recognise those common patterns in which mathematics is rich is a serious detriment to our capacity to appreciate its beauty.

The similarities between music and mathematics are well known and have provided, and undoubtedly will go on providing material for many essays. Let's not forget what the great Leibniz wrote: "Music is the pleasure the human mind experiences from counting, but does not know that it is counting." Here, I am going to content myself by indicating the important difference between music and mathematics. In the case of music, our ear automatically and sequentially makes the melody, the elements of rhythm, cadence, composition, etc. of a particular piece of music available to the brain, which will use this information to decide on the harmony of its various elements and on its beauty. But, which of our senses automatically makes the chain of mathematical ideas inherent to a great theory available to the brain?

The case of gastronomy

These considerations are also valid for more complex situations in which several senses are involved, such as in the case of gastronomy. Here, practically all the senses are involved, and therefore the sensory flow is more complex, but equally effective. All the senses are involved in tasting a wine. Starting with hearing, which transmits the sound of the wine pouring into the glass (transmitting information on the glycerin/alcohol balance); then sight, which communicates the tone, intensity and brightness of the colours; smell, which offers the brain a whole world of information camouflaged in the bouquet (generated by the variety of grape, the production process or the ageing conditions); taste, which transfers the distribution and balance of the five basic flavours; and also touch (as it passes through the mouth), which details the internal harmony of the various components of the liquid. All the senses make the brain aware of the organoleptic properties that help us to evaluate wine. In this example, we have to assume a few minimum starting conditions about

the individual for the aesthetic evaluation. First, there must not be any religious or moral commitments that may impede them in the evaluation. This, in some small measure, is also applicable to painting or sculpture. Think about the famous secret room in Madrid's Hapsburg palace where nude paintings were hidden, or the Nazi censoring of impressionist, expressionist or avant-garde artworks, which they classed as degenerate art. Secondly, there must be a certain level of culture and education of the palate, including the sense of smell, which translates into the capacity to appreciate, distinguish and recognise tastes and smells; regrettably they do not teach us about smells and tastes at school as they do with colours.

Olfactory memory allows recognition and comparison of the aromas that appear in wine. Of course, one should avoid the threat of the sensory atrophy that is produced by the tyranny of sticking to just a few tastes and smells. It does not take a huge effort of abstraction to transpose most of these impediments to the enjoyment of mathematical reasoning. Think of the burden of antipathy that mathematics imposes on most of the population and on the sensory atrophy that rejection can cause. Now is not the time to get into the issue of how the negative image of mathematics came to be, although here I will give one of the few possible causes as it may give you something to consider. Advertising has achieved the considerable miracle of converting a black, sweet drink into the real thing, simply by repeating the phrase billions and billions of times; the same thing has happened to mathematics, but in reverse.

Literature

Finally, I will consider an example that is linked much more closely to the case of mathematics: literature. In this case, it is sight (hearing, if we are being read to, or touch if we are reading in Braille) which communicates the various events that form a novel, or the different verses that make up a poem, to the brain. But compared with the automation with which the brain captures the pictorial elements of a painting, or the cadence and melody of a string quartet, now the brain must undertake a certain labour of analysis. This is due to the fact that the aesthetic object, in the case of literature, does not have visual, audio, olfactory or tactile perceptions; instead it is made up from meanings and these are not perceived sensorially. They are, on the contrary, a product of a fairly intense process of thought. The aesthetic value of a novel or a poem is not in the black and white of the text but in the meaning that is written there. We could say that literature has aesthetic value, but not perceptive value.

When five senses are not enough

The above examples illustrate and justify my initial statement: it is difficult to appreciate the beauty of mathematical reasoning because we do not have a sense suitable for capturing the configuration of the ideas where the beauty lies.

The same thing happens with mathematical reasoning as does with literature, it has aesthetic value, but not perceptive value. The appearance, the shape (in the Hegelian sense) of mathematical reasoning, which is what we appreciate sensorially, is irrelevant from the point of view of its aesthetic appreciation; that which matters resides in its content and its meaning. Consider that, although it serves to describe and explain reality, mathematics seems to happen in the heads of humans, such that a theorem is no more than a transfer from the brain of one individual to another (perhaps using the page of a book or a blackboard as an intermediary). Therefore, there is nothing more independent from the senses than mathematics.

So it is not surprising that the meaning of mathematics is not captured by an act of perception, but by a process of discursive thinking. In other words, mathematics is a repository of meaning and aesthetic value, but one which cannot be appreciated by means of the senses, but through a process of intellectual analysis. This is where the difficulty in perceiving beauty in mathematics compared to that of a painting, a sculpture or a musical composition lies. The effort required to create the series of mathematical ideas that make up a particular theorem is obviously not always equally complicated, and there are, in fact, certain resources that make it easier; resources that, once again, are linked to the senses. Think of the most common of these resources – using drawings and geometric figures to illustrate mathematical reasoning and ease its understanding. This is not surprising, as we are once again appealing to the automatism, immediacy and ease with which sight takes certain information to the brain.

But, although sight, hearing and touch put the statement of a theory or its demonstration within reach of the brain, the structure of ideas is not necessarily explicit in that demonstration. The structure of ideas is often hidden behind the logical reworking, analysis and explanation of theorems. It appears fractured and broken by intermediate steps and demonstrations of secondary facts that hide the path of ideas and impede us from appreciating its harmony. The capture of the structure of ideas by the brain is not automatic, but the fruit of study, an analysis of the elements that make up the proof, a draining of the dispensable, and a reordering and re-composition of the main ideas. As it is not an automatic

The automatism with which the senses transfer the shape, colours, composition, space, light and texture of a painting, or the harmony and rhythm of a musical composition, to the brain is broken in the case of mathematics. The labour of intellectual analysis is no longer automatic, it requires an effort and the greater the effort required, the more complicated and difficult the appreciation of the beauty of mathematics will be. However, the rewards may be more intense, as the brilliance, depth and genius of a great mathematical idea may be revealed by intellectual analysis.

Once the structure of ideas has been captured, it will generate, depending on the magnitude of the harmony of its components, a certain aesthetic pleasure, a spiritual delight, to quote the dictionary. By paraphrasing the philosopher George Santayana in the description of aesthetic value that he makes in his book *The Sense of Beauty*, it can be said that this objectified pleasure, this intellectual delight, which, if we are able to distil it, we get from studying and understanding a theorem, is the central aesthetic category, the quality of mathematical reasoning that gives it beauty.

The senses channel to the brain whatever is outside of it, therefore they are essential for enjoying the beauty of something which is outside of us, be it a painting, a symphony or a landscape. However, the satisfaction produced by beauty is not merely sensory, it also requires the intervention of reason. "The pleasures of beauty," wrote Fernando Savater, "are the least 'zoological' of all." Thus, it would be senseless to attempt to make a dog or a gorilla appreciate the aesthetic value of a Gothic cathedral or a painting by Velázquez, as much as their sight puts the harmony of the architectural or pictorial elements that they are made up of within reach of their brains. Philosophers such as Santayana defended the close link between aesthetic values and other values that are vital to the human being (although there are aspects that set them apart). Wittgenstein took that tendency to its limit when he established the equation: "Ethics and aesthetics are one." In any case, it is this link between beauty and reason that makes mathematics a depository of aesthetic value.

The crossed paths technique

As stated in the preface, the purpose of this book is not to weave a theoretical discourse on the aesthetic value of mathematics, but to illustrate and support some basic principles of the beauty of mathematics with well-selected examples. That is what

I propose to do next. We are already aware of the difficulty involved in managing to see the beauty encapsulated in a particular piece of mathematical reasoning – a difficulty that is similar, as previously stated, to the problem that literature has in showing its aesthetic value. However, in this case, as literature deals with the human condition, this difficulty is lessened: the emotional aspects of our condition are much closer and more accessible, and therefore more interesting to us, than the coldness of a right-angled triangle or the exoticism of a prime number. Mathematics, however, also has an emotional part, which is more intense and important than is imagined (and I am going to dedicate the entirety of the next chapter to this). So, as mathematicians we should be as capable of exploiting the emotional resources as novelists are and transfer some of the techniques used in novels, with all due precaution, to mathematics. Here I am going to focus on one of those techniques.

One of the main purposes of a novel – perhaps its main value – is to show the wealth, variety and complexity of the human condition. With the 20th century, one of the most stylistic resources for achieving this objective was born – depicting the human ants' nest that every large city inevitably ends up becoming and the vast tangle of humans that comprise it. Thus novels with a large number of characters began to appear, showing the exuberance and complexity of the continuous proliferation of the inhabitants of a great city. These characters live lives that are apparently unconnected, each one unrelated to the rest, but as soon as the narrator's scalpel slices open reality, a dense web of surprising connections and relationships appears between them. Some stand-out examples of this genre are *Manhattan Transfer* (1925) by the American John Dos Passos and *The Hive* (1951) by Spanish Nobel Prize winner Camilo José Cela (whose list of characters stretches to 296 imaginary characters and 50 real ones, although a significant number of them could be described as fleeting).

In mathematics, situations where several, apparently unrelated results turn out to be inextricably connected are quite common. Mathematics is a coherent whole, and often an applicable point of view, or a brilliant idea, allows results, which at first glance lack any unifying relationship whatsoever. Just as with novels of the type of *Manhattan Transfer* and *The Hive*, showing that wealth of hidden relationships is a resource that would allow the beauty of mathematics to become more explicit. It should not be forgotten, as Jorge Wagensberg advises us in his book *Intellectual Pleasures*, that finding the same principle in two different things is a paramount source of aesthetic satisfaction: "Realising that two things which are

theoretically different are, actually, one and the same is the basis of comprehension and of a rare intellectual pleasure.” I am going to dedicate the rest of this chapter to a detailed example of this.

Tangent circles, rational approximation, Diophantine equations and Cela's *The Hive*

From among the teeming swarm of results which populate the unfathomable geography of elemental mathematics, I am going to choose, as if at random, three. They will be the three characters of the story which I propose to tell; as in the pages of *The Hive*, these characters all seem to be as unrelated to one another as the vast expanse and variety of mathematics allows, although in the end that distance and alienation will turn out to be more than apparent.

The main character is a member of the old neighbourhood of geometry. It is those ever-evocative geometric constructions formed by tangent circles. Basically, I am going to give each one of these three characters a name. At this point I do not think the reader will find it too surprising that I am going to use the names of some characters from *The Hive* for this. So, I will call my first character Doña Rosa. In Cela's book, Doña Rosa is the owner of the La Delicia café, where a lot of the novel's scenes take place. “For Doña Rosa,” Cela explains, “her café is the world, and everything else revolves around the café. Some people claim that Doña Rosa's little eyes begin to sparkle when spring comes and the girls go in short sleeves. I think this is sheer gossip, for Doña Rosa would not sacrifice a solid five-peseta piece for anything in the world, spring or no spring. What Doña Rosa likes is simply to drag her great bulk about between the tables.”

The second character in my story lives in the nearby working-class suburb. It is about how well we can approximate any number: to give you an idea, think of $\sqrt{2}$ or the number π in terms of fractions. I am going to call this character Martín Marco. In Cela's novel, Marco is a poet, a left-wing idealist left on the sidelines since the Spanish Civil War: “He is a small man, stunted, pallid, and feeble, who wears spectacles framed in thin wire. His jacket is threadbare and his trousers frayed.” Martín Marco survives on the charity of his friends and old acquaintances, on the fried eggs prepared by his sister Filo now and then – behind her husband's back – and on the warmth of empty prostitutes' beds in the brothel run by an old friend of his mother's.

Finally, the third of my characters is an inhabitant of mathematics' most select and exclusive area – the theory of numbers; it is Diophantine equation

$$p^2 + q^2 + r^2 = 3 \cdot p \cdot q \cdot r,$$

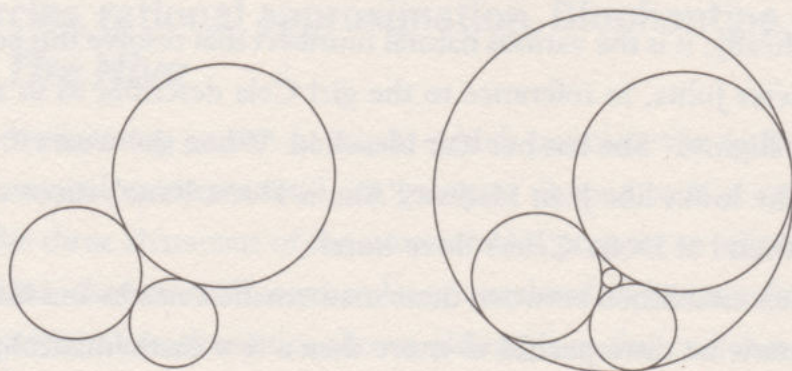
or more specifically, it is the various natural numbers that resolve this equation. I will call this character Julita, in reference to the girl Cela describes to us as rather fresh and a little bit flighty: “She has her hair bleached. When she wears it as a loose and wavy mane, she looks like Jean Harlow.” She is Doña Rosa’s niece and she meets with her boyfriend at Doña Celia’s ‘love hotel’.

The parallels established between these mathematical results and Cela’s characters may seem somewhat disrespectful to more than a few mathematical purists. I cannot deny that comparing geometry, or even a part of geometry, with the slobbish Doña Rosa, oafish, dirty and egocentric; or the rational approximation of irrational numbers with the gullible Martín Marco, a boastful loser; or a famous Diophantine equation with the posh Julita Moisés Leclerc, involves conceptual risks. But both the comparisons and their risks are a fundamental part of the metaphor that I propose to the reader. The biography of each of our characters goes back to the time of the Ancient Greeks, but as this story of crossed paths will end up showing, this will not be the only thing they have in common, or the most important. Although, just as with novels, in mathematics these similarities are not just coincidental.

Doña Rosa, or geometric constructions with tangent circles

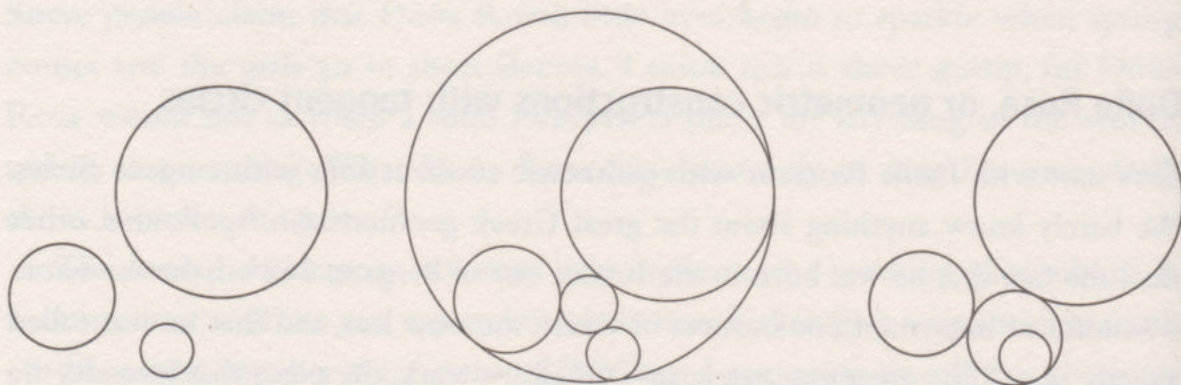
Let’s start with Doña Rosa, or with geometric constructions with tangent circles. We barely know anything about the great Greek geometrician Apollonius, other than the fact that he was born in the Ionian city of Perga in 262 BC; that he wrote a handful of important books, most of which are now lost, and that he was called exactly that: “The great geometrician”. Of all his work, the piece that interests me here is *Tangencies*. The original text is lost but we know of it because of what Pappus of Alexandria said in the 3rd or 4th century AD. In *Tangencies*, Apollonius solved what would later be called “the Apollonius problem”: given three elements, that may be points, straight lines or circles, draw a circle that is tangent to each of the three. Both the drawing and the number of solutions depend on whether we start with points, straight lines or circles, and the relative position of these elements. Apollonius, in any case, seems to have solved all possible cases.

When the three initial elements are circles, the first configurations with tangent circles appear. In short, if the three initial circles are also tangential, the problem has two solutions, one internal and one external.



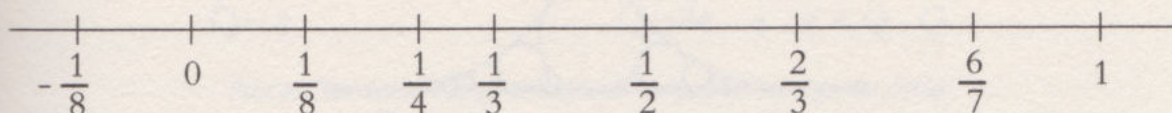
The Apollonius problem, when the initial configuration consists of three tangent circles (left), has two solutions (right).

The more general case where the three initial circles do not touch has, generally, eight different solutions.

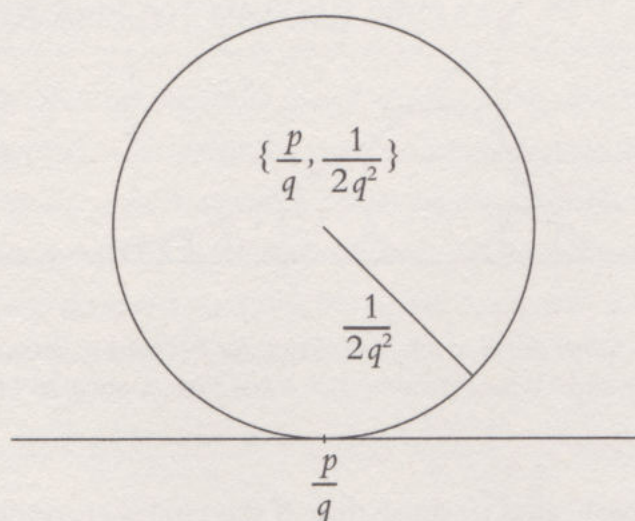


When the initial configuration consists of three circumferences that do not touch (left), the Apollonius problem has eight solutions (two of them are shown in the centre and another one on the right).

Of the many configurations of tangent circles that surround the history of mathematics, here I am going to consider an especially simple and elegant one: they are called Ford circles and they are constructed as follows. We start with a straight line on which we have marked the fractions (or the rational numbers, as we mathematicians like to call the fractions), just as shown below.

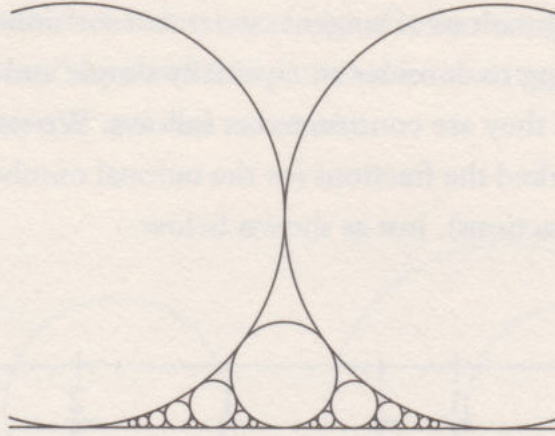


We are going to consider each fraction p/q irreducible – that is, neither p nor q have common divisors (for example, we reject the fraction $5/15$ in exchange for its irreducible equivalent $1/3$) and with a positive denominator q . On each fraction p/q we are going to place a circle that is tangential to the straight line with a radius of $1/(2q^2)$.



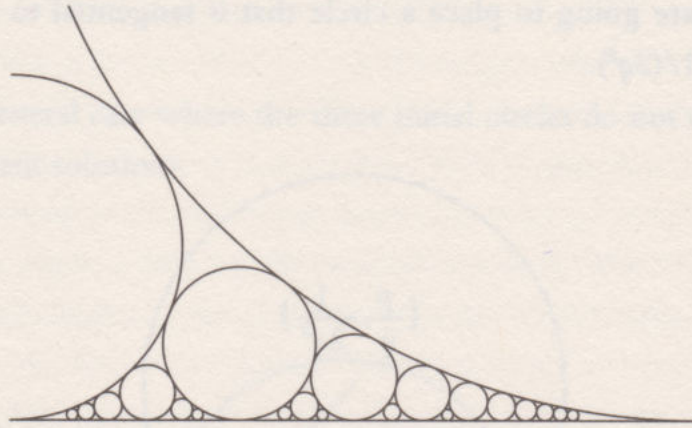
If we use the common Cartesian coordinate system to represent points on the plane (which the reader should have learnt at primary school), then the configuration of Ford circles constitutes all the circles the centres of which are on coordinate points $(p/q, 1/(2q^2))$ and whose radius is $1/(2q^2)$.

Ford circles have a lot of surprising properties. A relatively uncomplicated calculation shows that any two circles in this configuration either do not intersect or are tangential, just as we can see in the two diagrams overleaf.



Ford circles corresponding to the fractions between 0 and 1 whose denominator is less than or equal to 7. Thus, the circles correspond to the following fractions: 0, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 1.

The same calculation shows that the Ford circles corresponding to the fractions p/q and P/Q are tangential if numbers $p \cdot Q$ and $P \cdot q$ are consecutive – and vice versa.

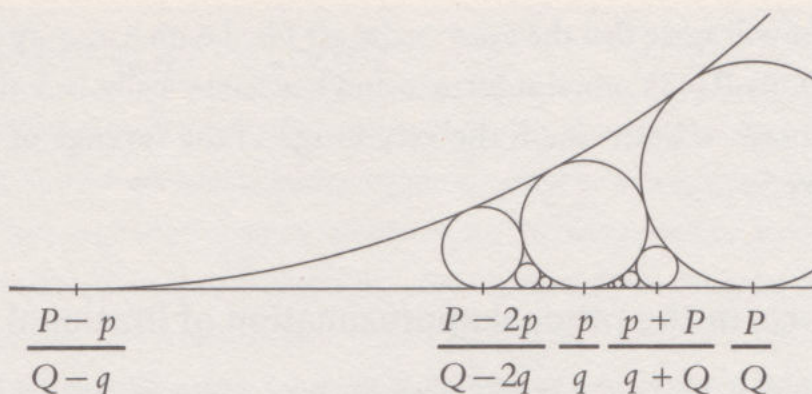


Another detail of the Ford circles: for fractions between 1/2 and 1 whose denominator is less than or equal to 11.

Nor is it too complicated to show that, if the circles corresponding to fractions p/q and P/Q are tangential, then the Ford circles corresponding to the fractions

$$\frac{P}{Q}, \frac{P+p}{Q+q}, \frac{P-p}{Q-q}, \frac{P+2 \cdot p}{Q+2 \cdot q}, \frac{P-2 \cdot p}{Q-2 \cdot q}, \frac{P+3 \cdot p}{Q+3 \cdot q}, \frac{P-3 \cdot p}{Q-3 \cdot q}, \frac{P+4 \cdot p}{Q+4 \cdot q}, \frac{P-4 \cdot p}{Q-4 \cdot q}, \text{ etc.,}$$

are also tangential to the circle corresponding to p/q . These fractions are also all those that correspond to Ford circles tangent to that of p/q .



Process for calculating Ford circles tangential to a given circle.

Equally, a simple calculation shows that Ford circles tangential to a given circle form a network that completely surrounds that circle. If we could draw all these infinite circles, we would see that inexorably they approach the fraction p/q until the point where they are chomping at it (see the diagram above and also the box below), as if they had the same voracious appetite with which Doña Rosa is described to us in Cela's novel.

THE VORACIOUS APPETITE OF FORD CIRCLES

The following simple calculation should be enough to convince you that Ford circles tangential to the circle corresponding to fraction p/q become increasingly closer to that fraction until they seemingly couldn't be any closer. Let's take a look at the tangent circles which are to the left of fraction p/q . These correspond to the fractions $(P+n \cdot p)/(Q+n \cdot q)$, where n is any natural number. All we have to do now is verify that the difference between these fractions and p/q becomes negligible as the number n increases:

$$\frac{P+n \cdot p}{Q+n \cdot q} - \frac{p}{q} = \frac{(P+n \cdot p) \cdot q - (Q+n \cdot q) \cdot p}{(Q+n \cdot q) \cdot q} = \frac{P \cdot q + n \cdot p \cdot q - Q \cdot p - n \cdot p \cdot q}{(Q+n \cdot q) \cdot q} = \frac{P \cdot q - Q \cdot p}{(Q+n \cdot q) \cdot q}.$$

So, as the circles corresponding to p/q and to P/Q are tangent, we get, according to the above, that the numbers $p \cdot Q$ and $P \cdot q$ are consecutive; their difference therefore must be equal to 1 or -1 . Finally, if we include this in the above calculation, we get:

$$\frac{P+n \cdot p}{Q+n \cdot q} - \frac{p}{q} = \frac{\pm 1}{(Q+n \cdot q) \cdot q}.$$

Thus, as the number n increases, as it is in the denominator, the difference between p/q and $(P+n \cdot p)/(Q+n \cdot q)$, becomes increasingly smaller until it equals zero when n is infinity.

The reader will agree that the Ford circles are filled with harmony and elegance as much as Doña Rosa's moral substance and her flabby belly lack these very attributes. Cela says of her: "She is the very image of the revenge of the well-fed on the hungry."

Martín Marco, or the rational approximation of irrational numbers

Let's leave Doña Rosa, or the Ford circles, for now, and consider the biography of the second of our characters – Martín Marco, or the rational approximation of irrational numbers.

The essential scientific belief of Pythagoras and the ancient Pythagoreans, the basis of their mathematics and their rational explanation of nature, was that the whole universe could be reduced down to numbers. And for the Pythagoreans there were no more numbers than 1, 2, 3, 4, 5, ... and the fractions that can be formed with them.

However, when they took on the simplest geometric operation of all, that of measuring line segments, their scientific scaffold collapsed. They measured the diagonal of a square with sides of 1 and found that its length was exactly $\sqrt{2}$. Their frustrations began when they realised that the value, $\sqrt{2}$, cannot be represented by a fraction (see box opposite). What could be simpler than measuring the diagonal of a square? Well not even this can be done precisely using the natural numbers and their fractions. Legend has it that Hippasus of Metapontum, the Pythagorean who could not hold his tongue and revealed the terrible secret outside the sect, was executed by drowning – thrown overboard into the sea so that he was perpetually hit by the waves: "As having revealed the unspeakable secret he became the creditor of the most terrible punishment, being dispossessed of his being and transported to nothingness, from whence he came."

So, it soon became clear in the history of mathematics that, as well as the numbers 1, 2, 3, 4, 5, etc. (which we use to count), and the fractions that can be formed with them, we need other more 'complicated' numbers. In order to establish a certain anthropological distinction between normal numbers and complicated numbers: numbers loaded with symbolism were used. Thus, we call 1, 2, 3, 4, 5, etc. natural numbers, while the fractions that we can form with them are called rational numbers; as opposed to $\sqrt{2}$, $\sqrt[3]{5}$, π , etc., which we call irrational numbers, as if this insulting label sought to warn us of a certain unwholesome characteristic of those numbers.

THE IRRATIONALITY OF ROOT 2

It is in results such as this where mathematics displays the extraordinary power of logical reasoning. As there are infinite fractions and we cannot prove every case, how can we be sure that there are none that give 2 when multiplied by themselves? By using a great Greek invention that revolutionised mathematics: demonstration. A demonstration is an applicable logical justification of a mathematical result. Starting with an evident fact and by means of a chain of reasoning, each link in which is deduced logically from the previous ones, leads to another fact that is not evident and the truth of which is thus established. The first demonstration that concerns us here is attributed to Pythagoras and is as follows. First note that all fractions have an irreducible equivalent with a numerator and a denominator without common divisors. If there is an irreducible fraction, let's call it p/q , which when multiplied by itself gives a result of 2 (that is, $p/q \cdot p/q = 2$), then $p \cdot p = 2 \cdot q \cdot q$. We will now see that this is impossible. If $p \cdot p = 2 \cdot q \cdot q$, we can deduce that $p \cdot p$ is an even number (it is twice the other number). Given that the square of an odd number is always odd, we can also deduce that p is an even number. So the number p is double another number, which we will call k (that is, $p = 2 \cdot k$). Substituting the above equation gives us $2 \cdot k \cdot 2 \cdot k = 2 \cdot q \cdot q$, or, equally, $2 \cdot k \cdot k = q \cdot q$; therefore, $q \cdot q$ is even, in which case q is even. But this is not possible, because if the fraction p/q is irreducible then the numerator and denominator cannot both be even numbers.

This quirk of irrational numbers becomes obvious when we ask the most innocent of questions about them: what is the value of $\sqrt{2}$?; what is the value of π ? An irrational number, by definition, cannot be described by a fraction. I can provide a fraction that differs from it by just one millionth, but that fraction would not be equal to the irrational number. Of course I could also give a fraction which differs from it by just a billionth, but that fraction would not be equal to the irrational number either. I could also set a significant difference and find a fraction that differs from the irrational number by less than this difference, but even so, that fraction would not equal the irrational number either, therefore I will need another fraction if I want to decrease the difference. It is like a Biblical curse: with the same cruel monotony with which the days unfold during the times of difficulty described in Cela's *The Hive*, one fraction is followed by another and then another and another one after than, and the last one may differ very little from the irrational number, almost zero, but it will not be equal to the irrational number, and I will have to look for another one if I want to get closer still.

To describe an irrational number we have no alternative but to turn to fairly precise rational approximations. And we would need an infinite number of them in order to achieve complete accuracy. Thus arose a new type of problem in mathematics – the rational approximation of irrational numbers.

One of the first mathematicians to contribute to the discipline was Archimedes, who obtained the most famous rational approximation for the best-known of the mathematical constants. By approximating the length of a circumference with that of a 96-sided polygon, he obtained the result that the number π is little more than a thousandth smaller than the fraction $22/7$. Many others have improved on this approximation throughout the centuries: Chinese mathematician Tsu Ch'ung-Chih found the fraction $355/113$, which only differs less than 3 ten-millionths from π (the fraction would be rediscovered by several European mathematicians at the end of the 16th century).



Stamps dedicated to Archimedes and Tsu Ch'ung-Chih, the two ancient mathematicians who provided the closest approximations to π .

Starting in the 17th century, a series of developments unleashed a genuine craze, almost an addiction, for calculating rational approximations of the number π , a craze in which distinguished mathematicians such as Newton and Euler participated.

But how well can we generally approximate an irrational number using fractions? Let's be a little more specific with the question. An irreducible fraction p/q is more 'expensive' to produce the greater its denominator q is (because to produce it the unit has to be divided into as many equal parts as indicated by the denominator). Therefore, in order to know how good or bad fraction p/q is as an approximation of an irrational number, we must compare the difference between the fraction

and irrational number in terms of the denominator q of the fraction. Given any number, let's call it a , it is a case of trying to estimate the lowest value of $|a - p/q|$ from all fractions p/q with a fixed denominator equal to q . Here I am using common mathematical notation to measure how different two numbers are. I write the difference between bars, $|x - y|$, to indicate that I always consider the difference from the largest number to the smallest (to obtain a positive result; specifically: $|x - y|$ has a value of $x - y$ if x is greater than y , while $|x - y|$ has a value of $y - x$ if y is greater than x).

Given that all the fractions whose denominators are equal to q are distributed along the straight line at an equal distance of $1/q$, we can conclude that whatever irrational number a is, there will always be a fraction p/q , such that $|a - p/q| < 1/(2 \cdot q)$. In other words, we can always approximate an irrational number with a fraction with an error smaller than the inverse of double the denominator.

For example, if we take the number π and $q = 10$, and we resort to the use of a calculator, we find that the best rational approximation of the number π with a denominator of 10 is the fraction $31/10$. In this case: $\pi - 31/10 = 0.04159\dots$, which indeed is less than $1/(2 \cdot 10) = 0.05$. And that is the best we can do with a denominator of 10. However, with other denominators things can be considerably improved. Take, for example, $q = 7$. The best rational approximation of the number π with a denominator of 7 is Archimedes' fraction $22/7$; in this case $|\pi - 22/7| = 0.00126\dots$; as you can see, Archimedes' fraction $22/7$ is better for approximating π than $31/10$ above (despite having a smaller denominator). Something similar happens if we use a denominator of 113; in this case the best approximation is Tsu Ch'ung-Chih's fraction $355/113$ $|\pi - 355/113| = 0.000000266\dots$. While with a denominator of 125, despite being larger than 113, the best we can do is $393/125$, which gives a considerably worse approximation: $|\pi - 393/125| = 0.0024\dots$ (even worse than Archimedes' fraction).

Given that there are denominators with which we can approximate an irrational number much better than with others, the question is not so much how well we can approximate with any denominator, but how well we can approximate with one which has been suitably selected.

With this in mind, German mathematician Lejeune Dirichlet (who married a sister of the composer Felix Mendelssohn) showed in 1842 that an irrational number can always be approximated by means of fractions with an error smaller than the inverse of the square of the denominator.



*German mathematician Peter Gustav Lejeune Dirichlet (1805–1859),
who ended up filling Gauss' chair in Göttingen
after the latter's death in 1855.*

The demonstration is quite basic and uses a principle made famous by Dirichlet – the pigeonhole principle. The pigeonhole principle is a simple expression of common sense. If we want to house a certain number of pigeons in cages and we have more pigeons than cages, then in the end there will be more than one pigeon in some cages. Despite its simplicity, the pigeonhole principle is a useful tool for demonstrating certain mathematical results; among which is Dirichlet's rational approximation – given any irrational number a , there are infinite fractions p/q such that $|a - p/q| < 1/q^2$ (the demonstration is included in the box opposite). This is a notable improvement on what we had previously, because as q increases the number $1/q^2$ becomes considerably smaller than $1/(2 \cdot q)$.

It is not possible to improve Dirichlet's result in terms of the power of two of $1/q$. This is intimately linked to the separation of irrational numbers in algebraic and transcendental numbers. Let's think about $\sqrt{2}$. It is an irrational number but, in a way, it is easy to describe it with our basic sequence of integer numbers: ..., -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6..., as $\sqrt{2}$ is the solution to the equa-

DIRICHLET AND THE PIGEONHOLE PRINCIPLE

Dirichlet's demonstration is as follows. Let's take any irrational number a , and a natural number N . Now let's consider the following numbers, $a, 2 \cdot a, 3 \cdot a, \dots, N \cdot a$ and $(N + 1) \cdot a$ (in total there are $N + 1$ numbers in this list). For each of them, we are going to generically call them $k \cdot a$, there will be a natural number p_k such that the difference $k \cdot a - p_k$ is between 0 and 1 (for example, if $a = \sqrt{5} = 2.236\dots$, then $2 \cdot a = 4.472\dots$, and p_2 will equal 4, $3 \cdot a = 6.708\dots$, and p_3 will equal 6, and so on). Now we separate the numbers between 0 and 1 into N different boxes. The first one will contain the numbers between 0 and $1/N$, the second, those which are between $1/N$ and $2/N$, and so on until the last one, which will contain the numbers which are between $(N - 1)/N$ and 1. Given that in our list of numbers $k \cdot a - p_k$, $k = 1, \dots, N + 1$, there are $N + 1$ numbers between 0 and 1, and we have divided the numbers between 0 and 1 into N different boxes, the pigeonhole principle establishes that one of those boxes contains more than one of our numbers. Let's say that the numbers $k \cdot a - p_k$ and $n \cdot a - p_n$ are in the same box. By construction, it is evident that the difference between two numbers in the same box is less than $1/N$; from which we can deduce that $|k \cdot a - p_k - (n \cdot a - p_n)| < 1/N$. Now, if $q = k - n$ and $p = p_k - p_n$, we get $|q \cdot a - p| < 1/N$, or, $|a - p/q| < 1/(q \cdot N)$. Given that both k and n are smaller than $N + 1$ we can deduce that q is smaller than N , and as that we can take it as positive we conclude that $|a - p/q| < 1/q^2$. As the number a is irrational and N is any natural number, the inequality $|a - p/q| < 1/(q \cdot N)$ confirms that there are infinite different fractions p/q verifying $|a - p/q| < 1/q^2$.

tion $x^2 - 2 = 0$, whose coefficients are integer numbers. We call numbers that are a solution to an equation whose coefficients are integers, whatever the degree of the equation, algebraic numbers. The number

$$\sqrt[4]{\frac{8}{\sqrt[3]{4 + \sqrt{\frac{557}{27}}} + \sqrt[3]{4 - \sqrt{\frac{557}{27}}}}} - \frac{\sqrt[3]{4 + \sqrt{\frac{557}{27}}} + \sqrt[3]{4 - \sqrt{\frac{557}{27}}}}{2} - \sqrt{\frac{\sqrt[3]{4 + \sqrt{\frac{557}{27}}} + \sqrt[3]{4 - \sqrt{\frac{557}{27}}}}{2}}$$

as monstrous as it may seem, is algebraic, because it is a solution to the equation of the fourth degree $x^4 + 8x - 5 = 0$, whose coefficients are all integers. Mathematicians describe numbers that are not algebraic as transcendental and, in a sense, they are as far from the natural numbers as possible.

The most famous mathematical constants tend to be transcendental numbers, including the number π and also the number e – a fact that was not demonstrated until the end of the 19th century. That the number π is transcendental has an impressive corollary: the circle cannot be squared; in other words, starting with a circle, a square with an equal area cannot be constructed with a ruler and a compass. Squaring the circle was a problem that Greek mathematicians obsessed over, and whose solution was not found until the end of the 19th century. If we compare resolving a mathematical problem with an athletics record, the square of the circle was a record that took more than two and a half millennia to be broken!

It turns out that algebraic numbers have a fundamental flaw when it comes to approximating them with fractions: there is no better way to do it than that stipulated in Dirichlet's result. More specifically, if we take any algebraic number a and a number k strictly greater than 2 ($k > 2$), then apart from a few exceptions (always with a finite number), the following will be true: $|a - p/q| > 1/q^k$.



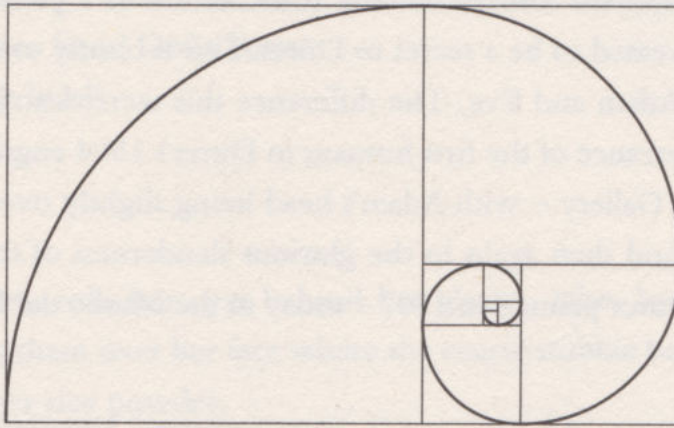
Adolf Hurwitz (1859–1919), one of the great mathematicians of the 19th century, particularly excelled in the fields of algebraic curves and the theory of numbers.

This lack of algebraic numbers tells us that Dirichlet's result cannot be improved in terms of the power of the denominator; but that which occurs with the constant that accompanies the square of the denominator is a different matter: 1. In 1891 another German mathematician named Adolf Hurwitz demonstrated that this constant could be substituted by a smaller one, $1/\sqrt{5}$. Thus, given any irrational number a , there are infinite fractions p/q such that $|a - p/q| < 1/(\sqrt{5} \cdot q^2)$. Hurwitz also showed that the constant, $1/\sqrt{5}$, cannot be improved on. The reason for this

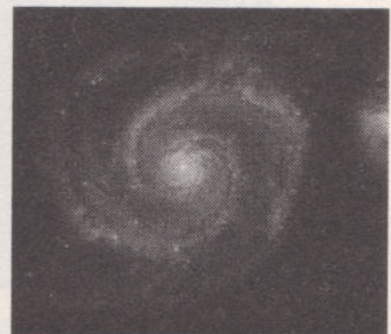
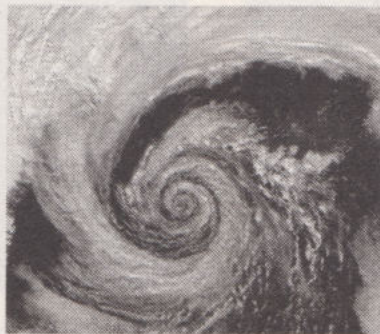
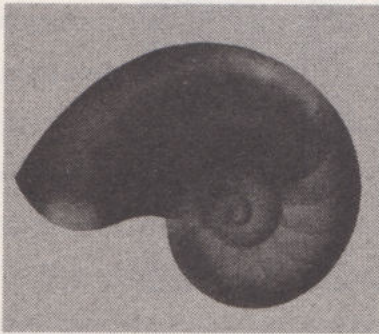
impossibility of improvement is another of mathematics' most celebrated constants – the golden number, $\Phi = (1 + \sqrt{5})/2$.

The golden number or golden ratio is the ration of the sides of a perfectly proportioned rectangle. According to the ancient Greek geometricians, the proportions of a rectangle are perfect when they are maintained in the rectangle obtained by eliminating a square with sides the same as the short side of the initial rectangle. Let's say that the dimensions of the rectangle are a , the short side and b , the long side. Then the dimensions of the new rectangle will be $b - a$ and a . We get the most harmonious proportion when $b/a = a/(b - a)$; which, if $x = b/a$, gives $x = 1/(x - 1)$; or, $x^2 - x - 1 = 0$. The positive root of this equation is the golden number: $\Phi = (1 + \sqrt{5})/2$.

If we apply the process of eliminating squares indefinitely, and join the opposite vertices of the squares with quarter circles, we get the golden spiral, as can be seen in the figure below.



This is the same shape as a nautilus shell, the arrangement of seeds in a sunflower, and the positions of the stars in many of the Universe's galaxies. The shape is also found in the cloud accumulations in depressions, cyclones and hurricanes.



The golden spiral is recognisable in nautiluses, hurricanes and galaxies.

The golden number is omnipresent in nature, and just as it has been an obsession for mathematicians, painters, architects, musicians and other artists have obsessed over it too. Dürer was a good example. Of all the Renaissance artists, he may have been the one who had the greatest knowledge of mathematics. That knowledge, which covered the fortification of cities, the use of the compass and the set square to measure solids, the study of human proportions and the geometric design of the alphabet, was recorded by Dürer at the end of his life in various texts, most of which were not published in print until after his death. Dürer acquired a good part of his mathematical knowledge in Italy. On the recommendation of the Venetian painter Jacopo de Barbari, Dürer visited Bologna in 1506 and there he was educated on the *secretissima scientia* by a teacher whose name he never revealed, although many believe they recognise the Franciscan friar Luca Pacioli in this anonymous figure. In 1494 Pacioli published the greatest mathematical encyclopaedia of the 15th century. From this point the *secretissima scientia*, behind which the golden number is hidden as an unknown formula for the construction of a perfect rendering for the human body, ceased to be a secret to Dürer. This is clearly evident in his magnificent nudes of Adam and Eve. The difference this secret knowledge made can be seen in the appearance of the first humans in Dürer's 1504 engravings – today in Vienna's Albertina Gallery – with Adam's head being slightly over-sized head and Eve a little tubby and then again in the glorious slenderness of the first couple in the oil-paintings Dürer painted in 1507 – today in the Museo del Prado in Madrid.



What did Dürer learn in the three years that separate the left-hand engraving from those on the right, which produced the appreciable difference in the bodily proportions of Adam and Eve?

As demonstrated by Hurwitz, the golden ratio is the irrational number which is the most difficult to approximate by means of fractions: Given any number $c > \sqrt{5}$ it can be shown that $|\Phi - p/q| > 1/(c \cdot q^2)$, apart from a few exceptions p/q (always a finite number).

Doña Rosa–Martín Marco, Ford–Dirichlet and Hurwitz

You would be pushed to find two characters in Cela's *The Hive*, who differ more in their way of life than Doña Rosa and Martín Marco. One fat and tired, a greedy misanthropist; the other squalid and hungry, wasteful (of what he has) and friendly. The two characters clash when Doña Rosa orders a waiter to throw Martín Marco out of her café for not paying for a drink. The owner gives instructions on what the waiter should do: "Into the street with the fellow, but gently, and when he's got him out there on the pavement, two good kicks where it hurts the most! He was taking the mickey!" However, the waiter holds back on violence, although he has no choice but to lie to Doña Rosa:

"Did you kick him?"

'Yes, madam.'

'How often?'

'Twice.'

The proprietress rolls her eyes behind her glasses, takes her hands from her pockets and passes them over her face where the coarse stubble begins to show, not quite hidden by her rice powder.

'Where did you kick him?'

'Where I could; on the legs.'

'Well done. That'll teach him! In future he won't try to steal decent people's good money.'"

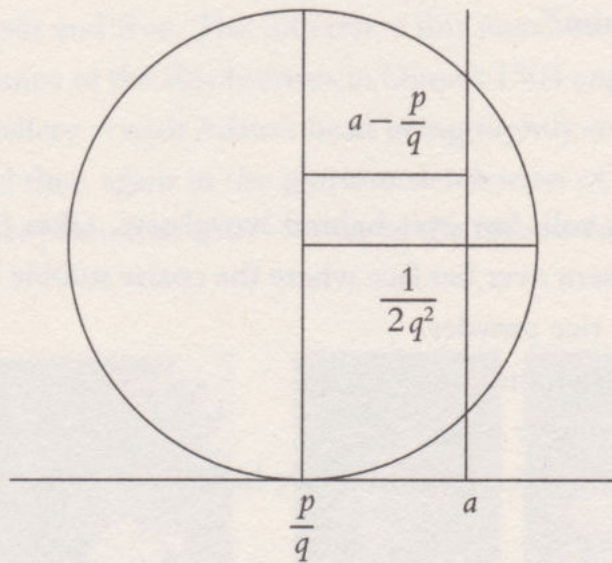
The configuration of Ford circles and the mildness of the approximation of irrational numbers by means of fractions that express Dirichlet and Hurwitz's theorems seem just as different as Doña Rosa and Martín Marco. Ford's circular grid, precise, elegant and harmonious; Dirichlet and Hurwitz's wheelings-and-dealings with fractions, arcane, shocking and strange. Two unrelated and different aspects of mathematics seem to conform.

But often in good novels the two unrelated and different characters turn into opposite, but complimentary, personifications of the same condition – that of our complex human nature. And no less frequently it turns out that two apparently dif-

ferent and isolated mathematical results, are none other than different expressions of one and the same specific mathematical reality.

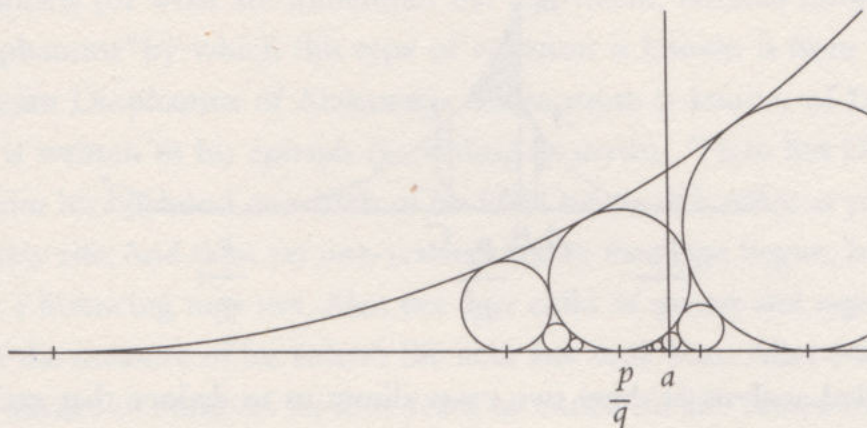
This is precisely what happens in Ford's configuration of tangent circles and the approximation of irrational numbers by means of fractions. The first is more than a geometric visualisation of the second, as if the farrago of Hurwitz's theorem had been crystalised in Ford's neat and tidy circles.

Take a look at the figure at the top of page 50. Although it may not look like it, what you see drawn there is none other than the visual materialisation of Hurwitz's theorem. Effectively, we draw irrational number a on the axis, and draw a straight line perpendicular to it on the point that represents the number, as shown in the figure below. Each time that straight line intersects a Ford circle (drawing the circle on rational number p/q), the geometry of the drawing tells us that the difference between a and p/q must be smaller than the radius of the circumference, that is, smaller than $1/(2 \cdot q^2)$: $|a - p/q| < 1/(2 \cdot q^2)$.



So, the fact that the Ford circles tangential to the circle corresponding to the fraction p/q conform to a network that inexorably approaches the fraction p/q to the point where it is chomping at it (see the diagram on page 51), allows us to see that if the straight line drawn on irrational number a intersects the Ford circle corresponding to fraction p/q , then it will also intersect another one which is tangential to it and located immediately below (see the diagram below); and, by the same reasoning, another one situated immediately below this one, and so on. From here we can deduce that the straight line drawn on irrational number a

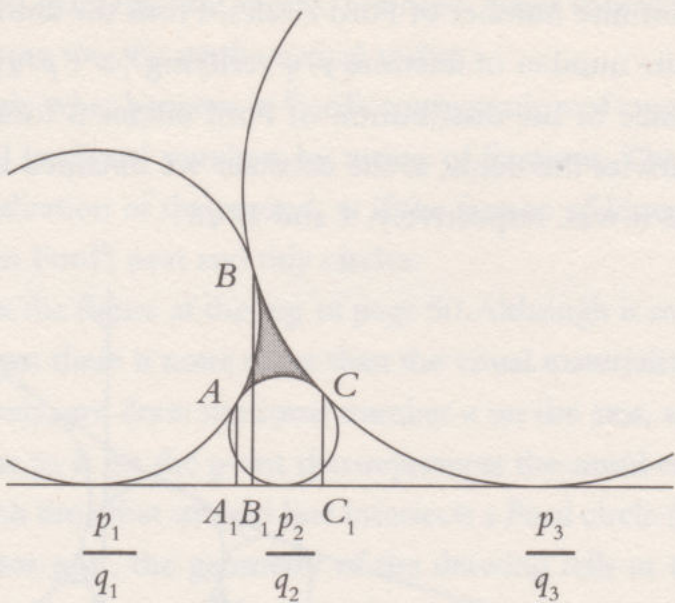
will intersect an infinite number of Ford circles. From the above, it follows that there are an infinite number of fractions p/q verifying $|a - p/q| < 1/(2 \cdot q^2)$. This strange consequence of the distribution of Ford circles is located between the Dirichlet and Hurwitz theorems, as the constant we obtained here is $1/2$, while in those theorems it was, respectively, 1 and $1/\sqrt{5}$.



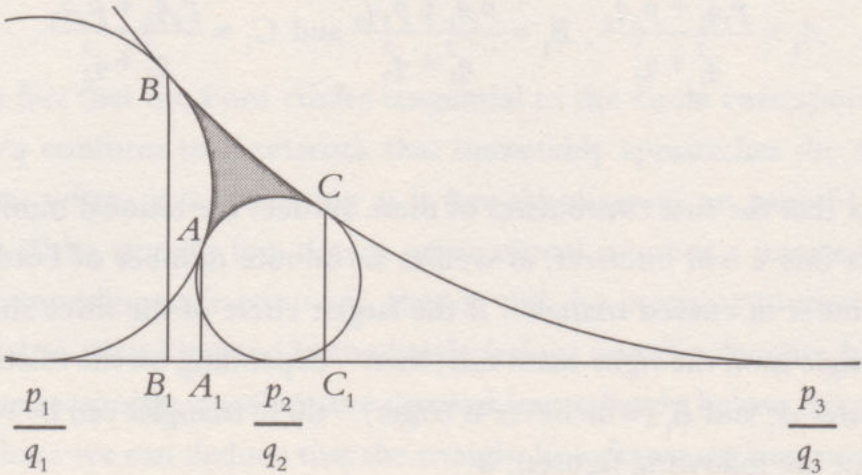
However, the Ford configuration is also able to predict the best of those constants, that which is given by the Hurwitz theorem. Effectively, in the Ford circles diagram (see the diagram at top of page 50) there are, as well as circles, other objects: triangles with curved edges between three tangent circles. These triangles also have very significant properties. One of them is that the first coordinate of its three vertices is, in all cases, a rational number. Let's consider the curved triangle formed by the Ford circles tangential to the corresponding fractions p_1/q_1 , p_2/q_2 and p_3/q_3 ; calling its vertices A , B and C . If A_1 is the first coordinate of the vertex A ; B_1 is the first coordinate of vertex B ; and C_1 is the first coordinate of vertex C , then it is not difficult to deduce that

$$A_1 = \frac{p_1 q_1 + p_2 q_2}{q_1^2 + q_2^2}, \quad B_1 = \frac{p_1 q_1 + p_3 q_3}{q_1^2 + q_3^2} \text{ and } C_1 = \frac{p_2 q_2 + p_3 q_3}{q_2^2 + q_3^2}.$$

The fact that the first coordinates of these vertices are rational numbers tells us that straight line a will intersect, as well as an infinite number of Ford circles, an infinite number of curved triangles. If the largest circle of the three that form the curved triangle is on the right-hand side, then – depending on the relative position of coordinates A_1 and B_1 (whichever is larger) – these triangles can be of two types (as shown in the following figures).



A detailed analysis of these two cases allows us to deduce that each time the straight line drawn on an irrational number a intersects the first type of curved triangle (when $A_1 < B_1$, see the diagram above), then the difference between a and p_2/q_2 is just less than $1/(\sqrt{5} \cdot q_2^2)$, while each time the straight line drawn on an irrational number a intersects the second type of curved triangle (when $A_1 > B_1$, diagram below), the difference between a and p_3/q_3 is just less than $1/(\sqrt{5} \cdot q_3^2)$. In all cases, the intersection of the straight line drawn on an irrational number a with curved triangles produces an infinite number of fractions p/q such that $|a - p/q| < 1/(\sqrt{5} \cdot q^2)$. Or, in other words, the arrangement of curved triangles generated by Ford circles is the geometric visualisation of the Hurwitz theorem.



Julita, or the Diophantine equation $p^2 + q^2 + r^2 = 3pqr$

But there was a third character in our story: the Diophantine equation $p^2 + q^2 + r^2 = 3 \cdot p \cdot q \cdot r$, who I named Julita, another character from *The Hive*.

A Diophantine equation is nothing more than an algebraic equation, normally with several variables, but of which we are only interested in solutions that are integer numbers (or what are sometimes the equivalent, rational numbers). The name ‘Diophantine’ by which this type of equation is known is from the Greek mathematician Diophantus of Alexandria. Little more is known of Diophantus than what is written in his epitaph (according to myth): “Here lies Diophantus, God gave him his boyhood one-sixth of his life; One twelfth more as youth while whiskers grew rife; And then yet one-seventh before marriage begun; In five years there came a bouncing new son. Alas, the dear child of master and sage, After attaining half the measure of his father’s life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.” Once resolved this tells us that Diophantus’ life lasted 85 years; a life that he supposedly lived between the 2nd and 3rd centuries AD.

We also know that Diophantus wrote various texts, the most important of which is *Arithmetica*, of which six of its thirteen books have been conserved in Greek and another four through Arabic translations.



The front cover of Diophantus’ *Arithmetica* from the 1621 edition created by French mathematician Claude Gaspard Bachet de Méziriac.

A DIOPHANTINE EQUATION

The problem described on this page is number 15 of Book II of *Arithmetica*. Diophantus finds the solution in the following way. He takes p and q as the squares of two consecutive numbers, because he already knew that their product added to their sum is also a square. Indeed, if $p = m^2$ and $q = (m + 1)^2$, then:

$$\begin{aligned} p \cdot q + p + q &= m^2 \cdot (m + 1)^2 + m^2 + (m + 1)^2 = m^4 + 2 \cdot m^3 + 4 \cdot m^2 + 2 \cdot m + 1 \\ &= (m^2 + m + 1)^2. \end{aligned}$$

In particular, Diophantus takes $p = 4$ and $q = 9$. In doing so he has now guaranteed that $p \cdot q + p + q$ is a square: $4 \cdot 9 + 4 + 9 = 7^2$. The other two quantities are now $4 \cdot n + 4 + n = 5 \cdot n + 4$ and $9 \cdot n + 9 + n = 10 \cdot n + 9$. He is now trying to find a number n such that both $10 \cdot n + 9$ and $5 \cdot n + 4$ are both perfect squares. He now introduces two new auxiliary variables r and k , defined by means of the equations $r^2 = 10 \cdot n + 9$ and $k^2 = 5 \cdot n + 4$. So we get

$$r^2 - k^2 = 10 \cdot n + 9 - 5 \cdot n - 4 = 5 \cdot n + 5,$$

which we can write as $(r + k) \cdot (r - k) = 5 \cdot (n + 1)$; now we do $r + k = 5$ and $r - k = n + 1$. Resolving for r and k we get: $r = (n/2) + 3$ and $k = 2 - (n/2)$. Substituting the value of r into the equation $r^2 = 10 \cdot n + 9$ and simplifying we get the quadratic equation $(n^2/4) - 7 \cdot n = 0$; resolving this gives $n = 28$.

Here is an example of the equations studied by Diophantus in *Arithmetica*: “Find three numbers such that the product of any two of them, increased by the sum of these two, is a square.” If we call the numbers p , q and n , the condition for them is that $p \cdot q + p + q$, $p \cdot n + p + n$ and $q \cdot n + q + n$ are square numbers. Diophantus provided solutions with the numbers $p = 4$, $q = 9$ and $n = 28$; and, effectively, $p \cdot q + p + q = 49 = 7^2$, $p \cdot n + p + n = 289 = 17^2$ and $q \cdot n + q + n = 144 = 12^2$ (see the box above). Diophantine equations had been studied in the Greek world long before Diophantus. This was probably the first: find natural numbers m and n such that $m^2 = 2 \cdot n^2$. As we have already seen, Pythagoras showed that the equation does not have a solution, since, as you may have noticed, for solutions to exist $\sqrt{2}$ would have to be a rational number.

Another Diophantine equation which was studied before Diophantus is related to Pythagoras’ theorem. Find all natural numbers p , q , r which are solutions to the

equation $p^2 + q^2 = r^2$; Pythagoras' theorem (or rather, its reciprocal) tells us that the numbers p, q, r would be the lengths of the sides of a right-angled triangle. These triples of numbers are known by the name Pythagorean triples. Book X of Euclid's *Elements* contains the more general solution: given any natural numbers m, n and k , then

$$p = k \cdot (m^2 - n^2), q = 2 \cdot k \cdot m \cdot n \text{ and } r = k \cdot (m^2 + n^2)$$

they form a Pythagorean triple – and they are all of this type. For example, taking $m = 3, n = 1$ and $k = 4$, we get $p = 32, q = 24$ and $r = 40$, which indeed solve the equation $p^2 + q^2 = r^2$.

Pythagorean triples were among the equations studied by Diophantus in *Arithmetica*. Diophantus also resolved the equation $p^2 + q^2 = r^2$ adding a multitude of additional conditions. For example, find the lengths of the sides of a right-angled triangle such that its perimeter is also a cube and the sum of its area and the hypotenuse, a square. The solution found by Diophantus was the hypotenuse r is $629/50$, and the other edges p and q are 2 and $621/50$; so the perimeter is $2 + 621/50 + 629/50 =$

ANOTHER DIOPHANTINE EQUATION

The problem last described on this page is number 17 of Book VI of *Arithmetica*. Diophantus finds the solution in the following way. He introduces a new variable n – the area of the triangle. So he has $(p \cdot q)/2 = n$, which is, $p \cdot q = 2 \cdot n$. Diophantus now takes $p = 2$ and $q = n$. The area plus the hypotenuse is therefore $n + r$ and the perimeter $2 + n + r$, as $n + r$ must be a square. We are trying to find a square such that it is a cube when 2 is added to it. Diophantus now places $m + 1$ on the side of the square and $m - 1$ on that of the cube. We are therefore looking for a number m such that $(m + 1)^2 + 2 = (m - 1)^3$. Or: $m^2 + 2 \cdot m + 3 = m^3 - 3 \cdot m^2 + 3 \cdot m - 1$. This is the same as: $4 \cdot m^2 + 4 = m^3 + m$, from which it can be deduced that $4 \cdot (m^2 + 1) = m \cdot (m^2 + 1)$, and therefore $m = 4$. So we get $n + r = 5^2 = 25$. As the triangle with sides p, q and r must be right angled: $4 + n^2 = r^2$. Substituting $n = 25 - r$ into that equation gives us $4 + (25 - r)^2 = r^2$; developing and simplifying gives: $629 - 50 \cdot r = 0$. Or, r is $629/50$; and, therefore, n and q are $621/50$.

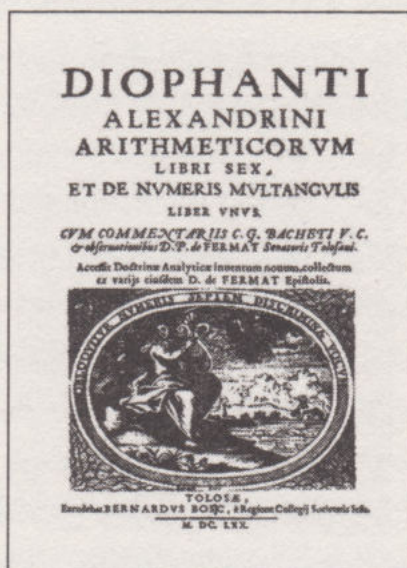
Note that Diophantus has found a solution with integer numbers for the cubic equation: $x^2 + 2 = y^3$ (more specifically: $x = 5, y = 3$). It turns out that this is the only integer solution allowed by this equation (although it has infinite fractionary solutions).

$1,350/50 = 27 = 3^3$, and the sum of the area and the hypotenuse is $(621/50) \cdot 2/2 + 629/50 = 1,250/50 = 25 = 5^2$ (see the box on the previous page).

In 1621, nearly a millennium and a half after Diophantus composed his *Arithmetica*, the six books conserved in Greek were printed in their original language. It is a commented edition made by the Frenchman Bachet de Méziriac, which also included a translation to Latin. There are few books that have gone down in history because of one of their readers, but this was one of them.

The reader was a French lawyer who answered to the name of Pierre de Fermat. But, as well as being a legal practitioner, Fermat was also an ‘amateur’ mathematician – although calling him an ‘amateur’ does not do him justice. Without doubt Fermat was a much better mathematician than most of those who, throughout the centuries, have dedicated themselves to mathematics professionally.

In the 17th century the theory of numbers was not the luxurious domain of mathematics that it would later become. After the splendour achieved by Diophantus, mathematicians’ interest in the theory of numbers had been languishing and dwindling for centuries, until its exclusive neighbourhood was practically abandoned. Then Fermat appeared on the scene and re-inhabited the theory of numbers by doing something which in mathematics has always been fruitful – proposing interesting problems. The notes and comments inspired by his reading Diophantus’ *Arithmetica* are evidence of this. In 1670, his son, Samuel de Fermat, compiled these notes and comments, added them to the Bachet de Méziriac edition and published them.



The cover of an edition of *Arithmetica* by Diophantus created by Fermat's son in 1670 which includes Pierre de Fermat's comments about what became his "last theorem".

There are not many of Diophantus' problems, or Méziriac's comments, to which Fermat did not propose an extension, a generalisation, or propose a mathematically interesting and profound question. Of all these problems the most famous was proposed regarding problem 8 of Book II: "Divide a square into a sum of two squares"; in other words, in this problem Diophantus explained how to generate Pythagorean triples: $p^2 + q^2 = r^2$.

As a great fan of mathematics, Fermat the lawyer slightly modified the equation and pondered the integer solutions of the equation $p^3 + q^3 = r^3$. Surprisingly he could not find any, apart from the so-called trivial ones, which are obtained by playing with the numbers 0, 1 and -1 . Given his lack of success, Fermat must have thought to himself, "What if I replace the exponent 3 with a 4? What would the integer solutions be to the equation $p^4 + q^4 = r^4$? His search for solutions was also unsuccessful in this case. "And what if I cannot find them because there are none?" Fermat would have asked himself after his failed searches. He then changed tack and searched for a demonstration that the equation with the exponent of 4 has no integer solutions. And, using an ingenious method of his invention, Fermat came up with the demonstration he was looking for. It is possible that he also modified this method to prove that the equation with an exponent of 3 has no solutions either, but we are not sure of this because, as he was not a professional mathematician, Fermat was never concerned with publishing the results he obtained, and even less so with the techniques and methods and the procedures that he used to obtain them. In many cases, all that we do know is practically guesswork.

Spurred on by his success with the equations with an exponent of 3 and 4 it is possible that Fermat thought that his method would work to demonstrate that for any number n greater than 2, the equation $p^n + q^n = r^n$ would not have any solutions either (apart from the trivial ones). Then what did he do? He wrote the following in the margin of his copy of Diophantus' *Arithmetica*: "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain." And that simple comment sent Fermat the lawyer headfirst into the history of mathematics, as from then, the world of mathematics went crazy searching for the "marvellous" proof that Fermat could not fit in the margin of Diophantus' book.

But Fermat's theorem turned out to be a very tough nut to crack, and in the ensuing centuries it was only proven for a handful of values of n : the prime numbers $n = 3$ (Euler, 1770), $n = 5$ (Legendre and Dirichlet, 1825) and $n = 7$ (Lamé, 1839);

and composite numbers $n = 6, 10$ and 14 . The complete proof for all cases did not arrive until 1994 and we owe it to Englishman Andrew Wiles. It contains several hundred pages and travels an unlikely route through some of the most sophisticated mathematical techniques and ideas developed in the 20th century.

Markov's equation

This Diophantine equation carries the name of the Russian mathematician Andrei Andreyevich Markov (1856–1922) and it is as follows:

$$p^2 + q^2 + r^2 = 3 \cdot p \cdot q \cdot r.$$

The natural numbers that resolve the equation (or, more specifically, the natural numbers p for which there are another two q and r , such that the three of them solve the equation), arranged in ascending order, are called Markov numbers. We know a lot of things about Markov numbers, although there are also others which we do not know. We know that there is an infinite number of them, and that the first 16 are:

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1,325, 1,597 and 2,897;

and there is also a simple procedure for constructing them, one after the other. It is as follows. It is very easy to show that if p_1, q_1 and r_1 solve Markov's equation and we write $p_2 = 3 \cdot q_1 \cdot r_1 - p_1$, $q_2 = 3 \cdot p_1 \cdot r_1 - q_1$ and $r_2 = 3 \cdot p_1 \cdot q_1 - r_1$, then the triplet p_2, q_1 and r_1 also solves Markov's equation. The same applies to the two triplets p_1, q_2 and r_1 , and p_1, q_1 and r_2 .

Markov showed that all the positive integer solutions to Markov's equation are obtained starting with $p_1 = 1, q_1 = 1$ and $r_1 = 1$ and applying this simple procedure successively.

It is surprising that Markov's equation has so many solutions, as it only has to be modified slightly for all of them to disappear; for example, the equation $p^2 + q^2 + r^2 = 2 \cdot p \cdot q \cdot r$ has no positive integer solutions. Actually, as demonstrated by Hurwitz, no equation of the type $p^2 + q^2 + r^2 = k \cdot p \cdot q \cdot r$ has positive integer solutions unless k equals 3 (Markov's equation) or 1.

Solutions p, q and r to Markov's equation with $p = 1$ establish the first connection with Hurwitz's theorem of rational approximation. Effectively, these solutions are of the type $p = 1, q = f_{2n-1}$ and $r = f_{2n+1}$, where f_k represents the corresponding

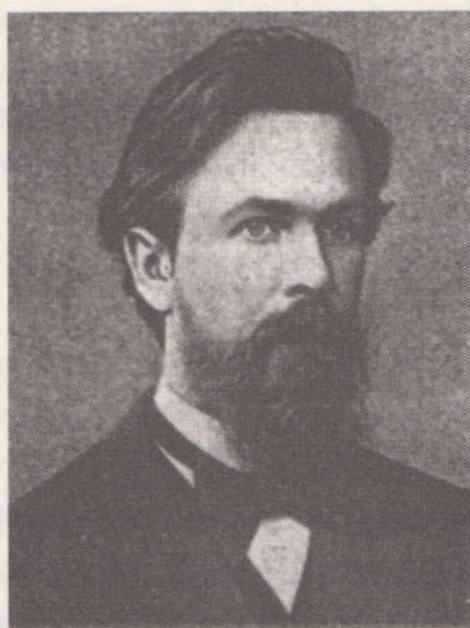
Fibonacci number. The first two Fibonacci numbers are $f_1 = 1$ and $f_2 = 1$, after which a Fibonacci number is obtained by adding the previous two together; this gives: $f_3 = 1 + 1 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21, f_9 = 34$, etc. Fibonacci numbers are as ubiquitous in nature as the golden number, which they are related to. Take the coefficient between two consecutive Fibonacci numbers, f_{n+1}/f_n , these fractions, $2/1, 3/2, 5/3, 8/5, 13/8 \dots$, approach the golden number in the proportion given by the Hurwitz theorem:

$$\left| \frac{1+\sqrt{5}}{2} - \frac{f_{n+1}}{f_n} \right| < \frac{1}{\sqrt{5} \cdot f_n^2}.$$

This establishes an undeniable connection between Markov numbers and rational approximation. Of course, the connection is much, much closer.

As mentioned before, the golden number is the reason why the Hurwitz theorem cannot be improved – actually the golden number Φ and all the irrational equivalents for the purposes of rational approximation. In other words, the irrational numbers of the type $(m \cdot \Phi + n)/(p \cdot \Phi + q)$, whatever the integers m, n, p, q , are, proving that $m \cdot q - n \cdot p = \pm 1$.

However, if we leave the golden number, and all of its irrational equivalents, to one side, then Hurwitz himself showed that his theorem leaves room for improvement, as the constant $1/\sqrt{5}$ can be substituted for a smaller one $1/\sqrt{8}$. Given any



Andrei Andreyevich Markov, a mathematician who made great progress in the fields of the theory of numbers and probabilities.

irrational number a , except the golden number and its equivalents, there are infinite fractions p/q such that

$$\left| a - \frac{p}{q} \right| < \frac{1}{\sqrt{8} \cdot q^2}.$$

This result, in turn, cannot be improved on; in this case because when $a = \sqrt{2}$, the approximation with fractions cannot be done any better than is allowed by the constant $1/\sqrt{8}$ multiplied by the inverse of the square of the denominator.

But if we leave $\sqrt{2}$, and all of its equivalents, aside, then we can continue to improve on it by substituting the constant $1/\sqrt{8}$ for another even smaller one, $5/\sqrt{221}$. Given any irrational number a , with the exception of the golden number, the square root of 2 and its respective equivalents, there are infinite fractions p/q such that

$$\left| a - \frac{p}{q} \right| < \frac{5}{\sqrt{221} \cdot q^2}.$$

As you have probably imagined, there is now another such irrational number that can impede the bettering of this constant. In this case it is $\sqrt{221}$, which will allow for a new improvement if it is eliminated: $13/\sqrt{1.517}$, which in turn has an undesired irrational quadratic that impedes improvement, an irrational number which, if eliminated, will produce a new improvement; so, continuing in this way, we would arrive at the limiting constant $1/3$. Given any irrational number a , with the exception of this list of irrational quadratics and their respective equivalents, there are infinite fractions p/q such that

$$\left| a - \frac{p}{q} \right| < \frac{1}{3 \cdot q^2}.$$

Now back to our novel analogy. Just as Martín Marco lives in the terror of the political repression of the Franco regime, Julita is suffocated by the Catholic morals that the same dictator and the church impose. But, just as for Marco there is no possible way out other than to conform, Julita and her boyfriend, both people of means, managed to overcome hurdles by turning to a brothel which they made

into their love nest. Cela painted the picture of moral contradictions masterfully. On the one hand, embodied in the character of Doña Visitación Leclerc, Julita's mother and Doña Rosa's sister, is the castrating ideology which was to the liking of the period's Catholic hierarchy (in all periods actually). Thus, we see Doña Visi piously sending money to the missionaries for the baptism of Chinese children, as a way of allowing them to enter into the supposed paradise with which God will compensate the righteous. On the other hand, Cela portrays in Julita's father, Don Roque Moisés, the slacker who lives well through marriage. Few scenes describe the national Catholic ethics of Francoism as well as the one where Julita and her father bump into each other the steps of Doña Celia's love hotel. Julita has just come from a passionate rendezvous with her boyfriend, while her father has gone there to meet one of his occasional lovers.

Just as the novel has the ability to highlight certain realities of the human condition, overlapping the paths of apparently unrelated lives, mathematics is capable of unveiling hidden truths in apparently unconnected results. This is the case with Markov numbers and those constants that appear in Hurwitz's theorem according to which we make improvements by eliminating the golden number, the square root of two and other irrational quadratics that impede further improvements.

Here are the first four Markov numbers (or solutions to the Diophantine equation $p^2 + q^2 + r^2 = 3 \cdot p \cdot q \cdot r$ in ascending order):

$$1, 2, 5, 13.$$

And here are the first four constants obtained by the successive improvements on Hurwitz's theorem on rational approximation:

$$1/\sqrt{5}, 1/\sqrt{8}, 5/\sqrt{221}, 13/\sqrt{1,517}.$$

Just as Martín Marco, Doña Rosa and Julita's lives are inextricably linked in the pages of *The Hive*, Markov numbers are inextricably linked to the rational approximation of irrational numbers, as they are the ones that determine the exceptions and the different constants that appear when improving Hurwitz's theorem.

Dear reader, the two lists of numbers above are actually one and the same, and to prove it all that is needed is the key that transforms one into the other. This key was found by the German mathematician Oskar Perron in 1921 and it is nothing more than the formula $m/\sqrt{9 \cdot m^2 - 4}$.

Indeed, if we use $m = 1$, the first Markov number in the formula, we get $1/\sqrt{9 \cdot 1 - 4} = 1/\sqrt{5}$, which is exactly the constant that appears in Hurwitz's theorem on rational approximation. If we now use $m = 2$, the second Markov number, we get $2/\sqrt{9 \cdot 4 - 4} = 2/\sqrt{32} = 1/\sqrt{8}$, which is exactly the constant we get in the improvement of Hurwitz's theorem by eliminating the golden number. And so on, if we use $m = 5$ or 13 , the third and fourth Markov numbers, we will get $5/\sqrt{221}$ and $13/\sqrt{1517}$, which are the next two constants that appear in the successive improvements on Hurwitz's theorem. And the same will happen with the following Markov numbers. On the other hand, if m , p and q are a solution to the Markov equation $m^2 + p^2 + q^2 = 3 \cdot m \cdot p \cdot q$ then the exception that impedes improvement of the constant $m/\sqrt{9 \cdot m^2 - 4}$ in Hurwitz's theorem is the number:

$$\frac{\sqrt{9 \cdot m^2 - 4} + m + \frac{2 \cdot p}{q}}{2 \cdot m},$$

and all of its irrational equivalents.

As you can see, in the republic of numbers, the lives of the characters overlap as much as in any large metropolis. In this case, as in many others, mathematics is more similar to the jumbled world of a hive than the cold logical structure that many assume it to be.

It would be unforgivable if the words of Cela did not provide the full stop to this chapter: "The morning unfolds slowly, it creeps like a caterpillar over the hearts of the men and women in the city; it beats, almost caressingly, against the newly awakened eyes, eyes which never once discover new horizons, new landscapes, new settings. And yet, this morning, this eternally repeated morning, has its little game changing the face of the city, of that tomb, that greased pole, that hive..."

Chapter 3

The Abstract and the Emotional: Mathematics and the Human Condition

Let's repeat our mental experiment of stopping someone at random on the street. This time we are going to ask them to do two things. First we will ask them to pair up the following words: literature and mathematics with passion and prudence; and then we will ask them their opinion on the human condition and mathematics.

In the first case, there will undoubtedly be more who associate literature with passion and mathematics with prudence.

Nor is there much doubt that our passer-by will respond to the second question that mathematics and the human condition are unrelated subjects. And, possibly, that would also be the answer of more than a few mathematicians. Mathematics is famed for being a set of abstractions that has little or no relation to the feelings of humans. But we should not forget that, almost more so than any other human creation, mathematics is born of our minds (in its most stark and solitary form). The logical structure of our brain is a fundamental characteristic of the human condition. Our brain is largely what makes us who we are.

So, as is often the case, it is possible that appearances could be deceiving us here. Firstly, remember that prudence, according to the dictionary, is "temperance, caution, moderation" and "soundness and good judgement"; while passion is "any disturbance or disorderly affections of the spirit" and "a vehement appetite or love for something". Despite the fact that many people do not relate passion with mathematics, it is also a battlefield in which prudence fights against passion. We mathematicians know that in our science there is an unstable balance between prudence and passion, that mathematics is a subtle mix of caution and vehement love, an intoxicating and disorderly affection of the spirit. Thus, when trying to demonstrate something, the mathematician is guided by the inherent prudence of the most strict logical thinking, but when undertaking a search for a discovery or struggling with a problem whose solution is unknown, the

mathematician succumbs to a certain unruliness of the spirit and an undeniably passionate intoxication.

Literature exploits the everlasting struggle between prudence and passion as a description of – yet at the same time also a source of knowledge of – human nature. So it is not surprising that most people associate literature with passion. But, again, mathematics and its circumstances have a lot to say about this clash between prudence and passion. It can, in fact, have a surprising use; to help us to reveal who we are and, therefore, also help us humans understand ourselves. In short, it helps us delve into the knowledge of the human condition.

Mathematics and its circumstances

I know that this sounds strange and that our imaginary passer-by would ask how mathematics, most of which they did not understand when taught it at school, can help them better understand mankind. And probably more than a few mathematicians, who understand the mysteries of their science, would not manage to see how it could illuminate the dark well that is human nature either. For the sceptics I must stress that mathematics possesses this illuminating capacity when coupled with its circumstances. By the circumstances of a theorem, for example, I mean the historical details in which its author, or authors, operated – be they the people who conjectured it, demonstrated or refuted it, or others who tried something or other without success, if that was the case.

My understanding of the circumstances of mathematics is that they are, in a way, similar to the circumstances described by Ortega and Gasset as inseparable colleagues for understanding one's self. These circumstances of mathematics have a lot to do with the history of mathematics, but, in my understanding, they are not one in the same.

Now I can better pin down this statement, which was implausible to our passer-by and doubtful to our disbelieving mathematician – namely mathematics can help us to better understand who we are. The confrontation between the abstract world of mathematics and the emotional world where those who discover or invent mathematics reside, gives out a light that can help to illuminate the innermost depths of human nature.

This is why the circumstances of mathematics better prepare us to appreciate its beauty. As explained in Chapter 2, the main difference between literature and mathematics, when it comes to appreciating their respective aesthetic values, derives from

the different entities that each discipline deals with. Literature deals with feelings, with the emotional part of the human condition, while mathematics deals with triangles, numbers and abstract objects. The greater closeness to, and knowledge we have of, our emotions and feelings allows us to better appreciate the aesthetic value of a novel, while the cold and abstract nature of mathematical objects makes it difficult. Hence the importance of the emotional circumstances that also belong to mathematics. With them we humanise mathematics itself, which, just as with literature, predisposes the spirit for aesthetic enjoyment.

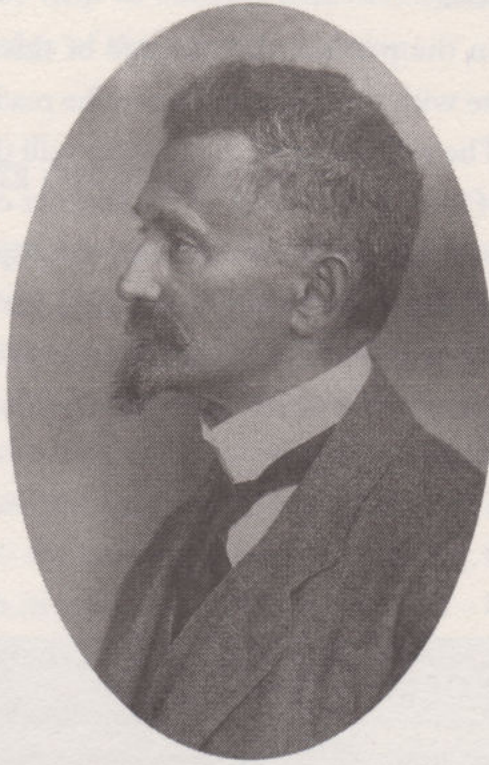
But, just as I stated in the preface, the purpose of this book is not to get into arguments but to illustrate with examples and leave the reader to subsequently draw their own conclusions. The remainder of the chapter will therefore be dedicated to presenting an example of how the contrast of the abstract character of mathematics with the emotions with which it has been created predisposes us to appreciate of its beauty – as well as helping us to better understand the human condition. And the example I have chosen is one of mathematical objects with undeniable beauty: fractals. The emotional circumstances relate to the mathematician who anticipated them – Felix Hausdorff (1868–1942).

THE MOST ANCIENT OF SCIENCES

As I have already said, I will not discuss, or delve into, a few particularly suggestive circumstances that link mathematics to the emotional part of the human condition, and which go back to the beginnings of humanity's mathematical vocation. These beginnings commenced with the expression of numbers, and it should not be forgotten that numbers awaited us at the end of our hands, residing in our fingers as if they were a part of our anatomy, nor should we stop appreciating the extent to which our hands have influenced what we are as a species. In the mist of prehistory, it is difficult to make out what we learnt first, whether it was marking numbers with our fingers, painting on the walls of caves, burying the dead or inventing gods and religions; all activities, including the numerical one, in which passion and prudence have fought, and will continue to fight, ferociously. This, in any case, makes mathematics the most ancient science, and means that its roots are buried in the most remote antiquity of *Homo sapiens*.

Fractals and the Hausdorff dimension

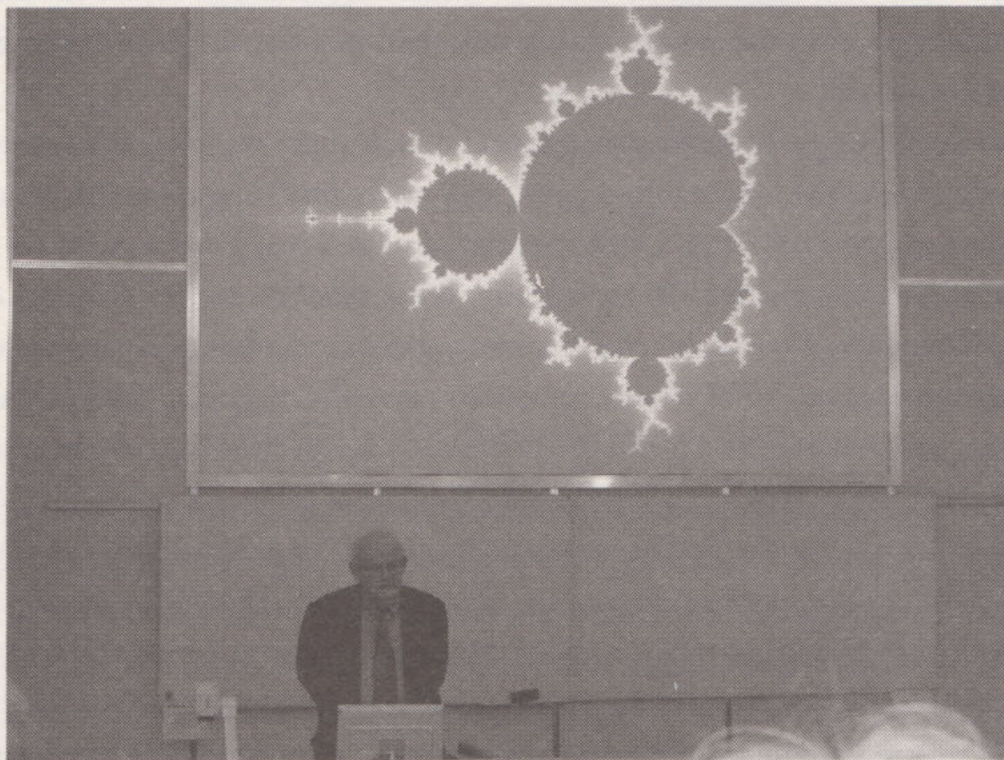
In a manner of speaking, a fractal is a set that our senses find unusual, although this anomaly is related to the way in which we perceive space. The strange thing about fractals is, therefore, sensory and not mathematical. The concept of spatial dimension lies in this anomaly, it is a concept that the German mathematician Felix Hausdorff significantly enriched in 1919.



The contributions of German mathematician Felix Hausdorff allowed the subsequent development of the modern theory of fractals.

Hausdorff thought that the traditional way in which, both mathematically and philosophically, the dimensions of an object were understood was poor, and he found the corresponding classification of the solids by their dimension very simple. Hausdorff said that it was very unsophisticated, and even incorrect, to say that an object has dimension 1 if it only has length (a thread, or a spring, for example); dimension 2 if it has width as well as length (a sheet of paper or the surface of a sphere); and dimension 3 if as well as length and width it also has height (a sphere or a shoebox). In order to enrich the traditional concept of dimension, Hausdorff proposed a notion that was, from a mathematical point of view, more sophisticated.

Hausdorff's notion allows the dimension of an object to be measured much more precisely; meaning that contrary to what intuition tells us, there are objects with fractional dimensions, dimensions $1/2$, or even, $\sqrt{5}$, or with even stranger numbers. More than half a century after Hausdorff detailed his new concept of dimension, Benoît Mandelbrot (1924–2010), a Polish-born mathematician who grew up in France and emigrated to America, named sets whose Hausdorff dimension is fractional fractals.



Benoît Mandelbrot, the mathematician who coined the term fractal, speaking at a conference held in Warsaw in 2005.

The Hausdorff dimension explained in its most general terms (just as he did) requires a certain level of mathematical knowledge. However, there is an alternative version which, although it foregoes a little accuracy, will allow the reader to obtain a reasonable understanding of what it consists of. The alternative way of defining dimension is owed to two Russian mathematicians, Pontryagin and Schnirelmann. It never ceases to amaze that one of them, Pontryagin, was blind (he lost his sight when he was 14 years old because of an explosion).

Imagine that we have a flat figure within a square whose Hausdorff dimension we want to calculate. To do so, we divide the side of the square into equal parts – let's say ten parts to start with. This divides the square into 100 smaller squares.

Now we count how many of the smaller squares are needed to cover our figure and that number is adequately compared with the number of parts into which we have divided the side, 10 in our case.

The key to this is the meaning that we attribute to “adequately compare”. Let me explain with a simple example. Our figure is going to be the complete square. It is obvious that to cover it we need all the smaller squares into which we have divided the square. So if we divide the side into a generic number n of equal parts, in total there will be $n \cdot n = n^2$ smaller squares. The reader should note the 2 that appears in the power n^2 , because that number, 2, is the exact dimension of the square.

Now let's change the geometric figure and look at the diagonal of a square. We divide the side of the square into 4 parts, how many smaller squares will be needed to cover the diagonal? After a little thinking you will reach the conclusion that 4 smaller squares are needed. To be specific, the 4 that are on the diagonal of the square. Therefore, if we divide the side into a generic number of n parts we would need an equal number n of smaller squares to cover the diagonal. But n is the same as n^1 (that is, n elevated to 1), and the power of 1 is, specifically, the dimension of the diagonal, because as it is a line segment we know its dimension is 1.

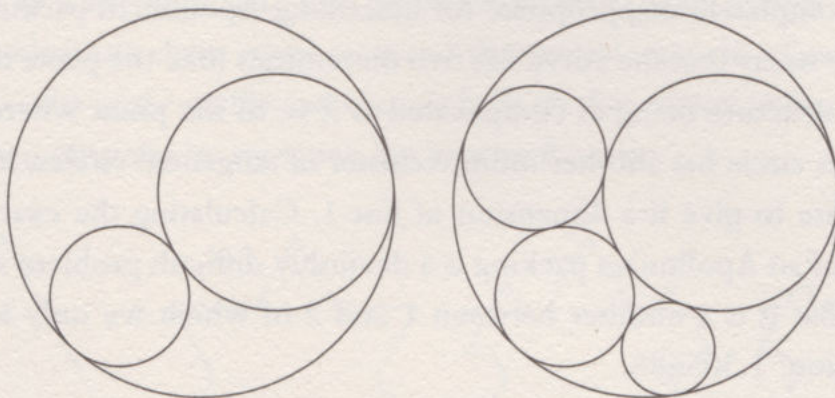
Now let's call the flat figure contained inside the square, whose Hausdorff dimension we would like to calculate, F . Once we have divided the square into n parts we count how many of the smaller squares generated are needed to cover our figure F . Let's call this number n_F . To adequately compare this number n_F with the number of parts n into which we have divided the side means to know to which power of n that number corresponds, n_F . Thus, in the case of the square, $n_F = n^2$, and it corresponds to the power of 2; and in the case of the diagonal, $n_F = n^1$, and it corresponds to the power of 1. If we call this power d , the relationship that links n , n_F and d will be $n_F = n^d$. Using logarithms we can express d in terms of n and n_F — d is the logarithm of n_F divided into the logarithm of n :

$$d = \frac{\log(n_F)}{\log(n)}.$$

The greater n is (the number of parts we divide the side of the square into) the closer this number d will be to the shapes's Hausdorff dimension F . This dimension is obtained in when we divide the side of the square into an infinite number of infinitesimally small equal parts.

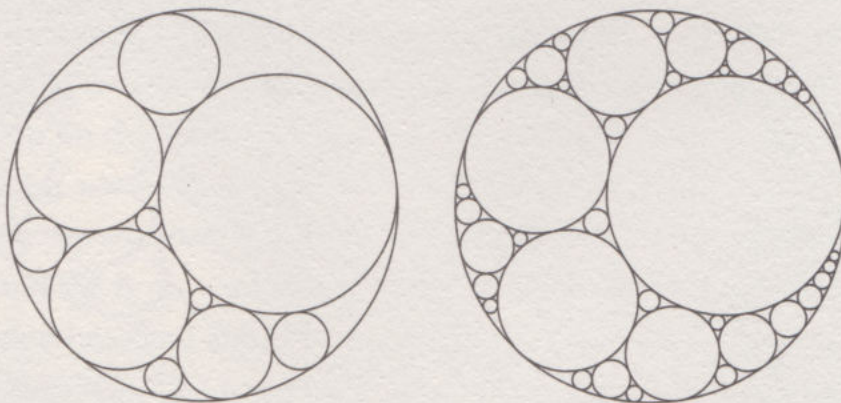
An example using circles of Apollonius

I will now build an example of a fractal. To do so I am going to go back to the circles of Apollonius, which appeared in Chapter 2, as we are going to build this fractal using tangent circles. We will start with three mutually tangent circles (below left). As we said in the previous chapter, there are two other circles tangential to these three. We now have five circles (below right).



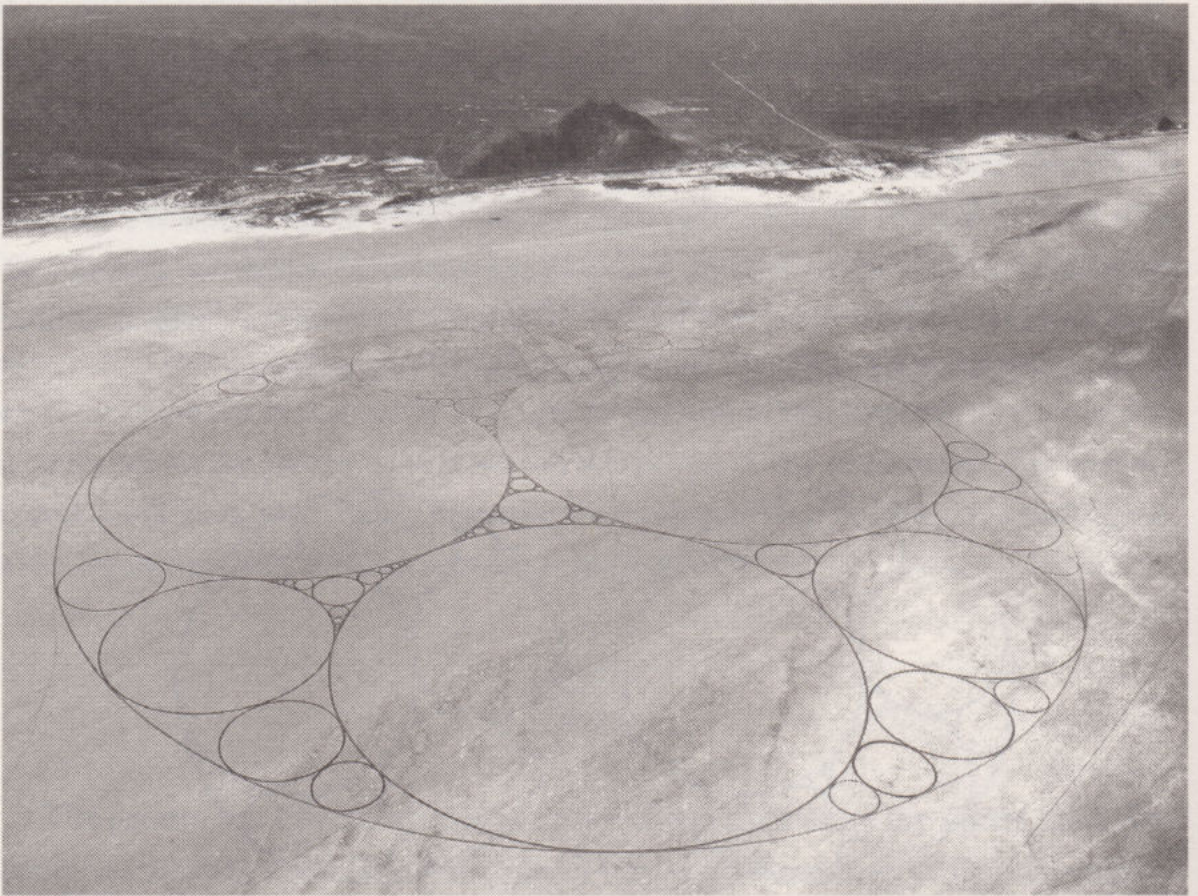
Construction of a fractal from three tangent circles.

We will use any three circles that are mutually tangential and generate the two corresponding tangent circles demonstrated by Apollonius' theorem. At the end of the process, and because there are repetitions, we will get six new circles. Together with the five we already have, that gives eleven circles (below left). We reapply the process to these eleven circles, and then again to those which we obtain, and so on and so on (below right) into infinity. The resulting family of circles is what is known as 'Apollonian packing', and it is an example of a fractal set.



Construction of a fractal from three tangent circles.

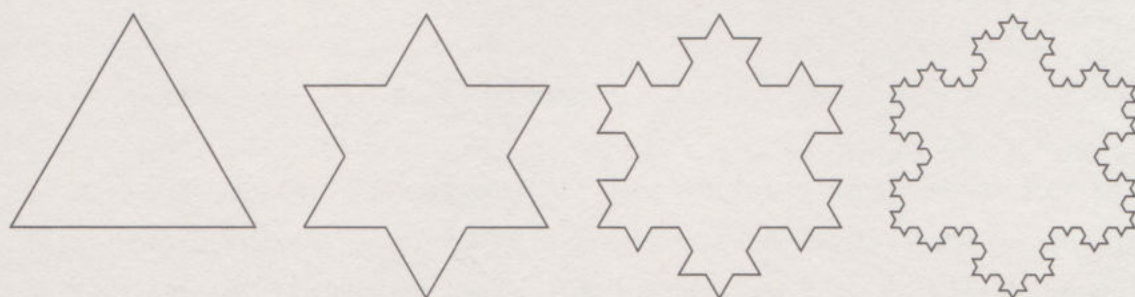
It is difficult to imagine the complexity of Apollonian packing, and that it could arise from a mechanism as simple as repeatedly drawing tangent circles. With a small stretch of your imagination you will be able to see that any of the circles in the packing is surrounded by an infinite constellation of other infinite tangential circumferences (except for the external one, which contains all the rest). Moreover, on any arc – however small it might be – of a circle in the packing, there is another infinite cluster of tangential circles. The standard notion of dimension is, therefore, completely inappropriate for describing Apollonian packing. It would be excessive to say that the curve has two dimensions (like the plane that contains it), but the structure being as complicated as it is, to the point where every part of each of its circles has another infinite cluster of tangential circles, it would also be inadequate to give it a dimension of just 1. Calculating the exact Hausdorff dimension of an Apollonian packing is a devilishly difficult problem and, to date, we know that it is a number between 1 and 2 of which we only have a good approximation: 1.305688.



Apollonian packing (of a truly colossal size) created by the artist Jim Denevan in the Nevada desert.

An example starting with a triangle

I will now construct another fractal, the Hausdorff dimension of which we are going to calculate exactly. It is the Koch curve, so called in honour of Swedish mathematician Niels von Koch, who defined it in 1906. There are different variations; here I am going to build the Koch curve starting with an equilateral triangle. To do this each of its sides is divided into three equal parts and the central segment of each is replaced by the two sides of the equilateral triangle built on that central segment. This gives us a six-pointed star. The process is repeated again – each of the twelve sides is divided into three equal parts and the central segment of each is replaced by the two sides of the equilateral triangle built on that central segment. The Koch curve is then obtained by repeating the process for ever.



The first four steps in the construction of the Koch curve.

Now imagine that the Koch curve is a road. Take any two points on the curve, as if they were two villages on the road. We get into an imaginary car and drive from one village to the next following the curve. What do you think the milometer will read at the end of the journey? If you think it depends on the start and end points selected, you will be wrong. It does not matter which points you choose, the mileometer will always read the same – infinity. In other words, any section of the Koch curve has an infinite length, because the curve twists and turns, it is so full of corners that it becomes a never-ending road (see box overleaf). This is similar to the situation on the west coast of Scotland. The direct distance from Fort William to Ullapool is a little more than 120 kilometres. But if you travel from one to the other along the coastline, the route becomes almost endless, as the path twists and turns around the firths, sea lochs and peninsulas. The few hundred metres of water between one headland and the next becomes a dozen kilometres or more when following the coast. The route becomes endless – or at least seems like it. The same effect, only infinitely exaggerated, is what happens when you try to travel a section of the Koch curve.

THE LENGTH OF THE KOCH CURVE

In order to convince us that the Koch curve has infinite length, we can do the following. We start by observing that in each step of the construction of the Koch curve we multiply the number of segments which it is composed of, as each one of the segments from the previous step is divided into three and we substitute one of them for two: in other words, where there was one segment, now there are four. As we start with the three sides of an equilateral triangle, the total number of segments in generic step N will be $3 \cdot 4^N$. For the same reason, the length of each one of these segments (they are all the same length) is divided by three in each step, therefore in generic step N that length will be $l/3^N$, where l is the length of the side of the initial equilateral triangle. So, the length of the curve obtained in generic step N will be the number of segments which form it multiplied by their length:

$$\frac{3 \cdot 4^N \cdot l}{3^N} = 3 \cdot l \cdot \left(\frac{4}{3}\right)^N.$$

As $4/3$ is a larger number than 1, the power $(4/3)^N$ it will become unimaginably large as the number of steps N increases in the construction of the Koch curve, therefore its length will be infinite. Similarly, we can convince ourselves that, actually, the length of any part of the Koch curve is infinite.

Just as with Apollonian packing, the standard notion of dimension is completely inappropriate for describing the Koch curve. It would be too much to say that the curve has a two dimensions (like the plane that contains it). But with the curve twisting and turning as it does, to the point that the length of any section of it becomes infinite, it would be too mean to say that its dimension is 1. Instead, the Hausdorff dimension gives us a comprehensive idea of where the Koch curve lies between being a curve and a surface. The value of this dimension is $\log 4 / \log 3$ (see the box opposite).

Mandelbrot showed that the geometry of fractals tends to be spectacularly complex (it is not unusual for them to be associated with processes of chaos), although this complexity often arises paradoxically from a simple similarity (on any scale) between its parts.

Fractals are strange, fascinating and labyrinthine sets, which are 'apparently' far removed from our physical experience. I have put 'apparently' in inverted commas

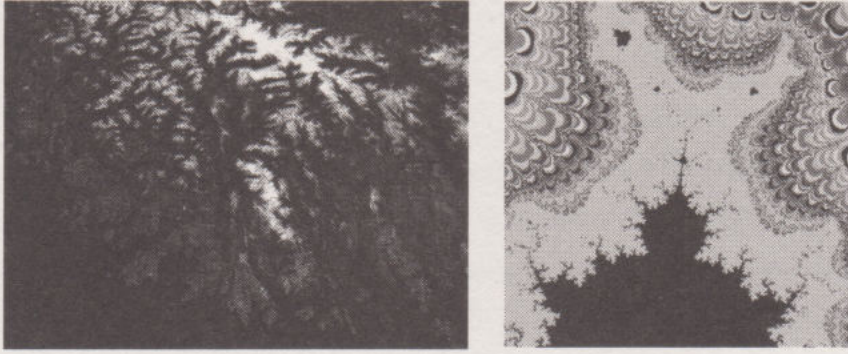
FRactal Dimension of the Koch Curve

The calculation of the fractal dimension of the Koch curve is relatively simple. Remember that the total number of segments in generic step N of the construction is $3 \cdot 4^N$, and that the length of each of these segments is $l/3^N$ (see previous text box). Taking into account its construction, we insert the curve into a square with sides of l (l is the length of the side of the initial triangle), and we divide it into an equal number of parts which is a power of the number 3: first into three parts, then into $3 \cdot 3 = 3^2$ parts, then into $3 \cdot 3 \cdot 3 = 3^3$, and so on. Now we count how many of the small squares we will need to cover the Koch curve when we have subdivided the side of the square into, let's say, 3^N parts. To do so we substitute the Koch curve for the curve obtained in step N of the construction process. As the length of the small squares is $l/3^N$, each one will cover, more or less, one of the segments which form the curve, the length of which is also be $l/3^N$; as we have $3 \cdot 4^N$ segments, we will need approximately $3 \cdot 4^N$ small squares. According to Hausdorff's definition, we have divided the side of the square into $n = 3^N$ parts and we need $n_f = 3 \cdot 4^N$ small squares to cover the curve; finally, we use the properties of logarithms to simplify the quotient which defines Hausdorff's dimension:

$$\frac{\log(n_f)}{\log(n)} = \frac{\log(3 \cdot 4^N)}{\log(3^N)} = \frac{\log 3 + N \cdot \log 4}{N \cdot \log 3} = \frac{1}{N} + \frac{\log 4}{\log 3}.$$

And when the number n of parts into which we divide the square, or N , which is the same, becomes infinite, it can be seen that the Hausdorff dimension of the Koch curve is $\log 4 / \log 3$.

because, actually, these objects are so omnipresent, they can be found before our eyes with such frequency, and we are so used to their strangeness, that we are not even capable of recognising them. Fractal geometry is more abundant in nature than the smooth geometry of the more conventional mathematical surfaces. For example, there is no better description of a coast riddled with lochs or fjords, such as the Scottish or Norwegian one, than a fractal curve like the Koch curve. And, equally, nothing better describes the network of neurones of our complicated brain than a fractal. And it is mathematical perspective (the acute vision of mathematicians like Hausdorff and Mandelbrot) which has allowed us to recognise the ubiquity of fractals in nature.



Fractals are not only a domain of mathematics, but they are also present in nature. On the left, an aerial view of the Norwegian fjords and, on the right, a fragment of one of Mandelbrot's fractals.

FRACTALS IN POETRY

As well as mathematics, the keen eye of poetry has also been able to detect the omnipresence of fractals in nature. Of the countless examples with which I could illustrate this coincidence of the poetic and mathematical vision of reality, I have chosen the first verses of Poem 18 of *Twenty Poems of Love and One Desperate Song* by Pablo Neruda. To capture the unreality of long distance love, "I love what I don't have. You are so far away", Neruda employed objects in this poem whose ethereal nature contrasts with the solidity of their three-dimensional constitution:

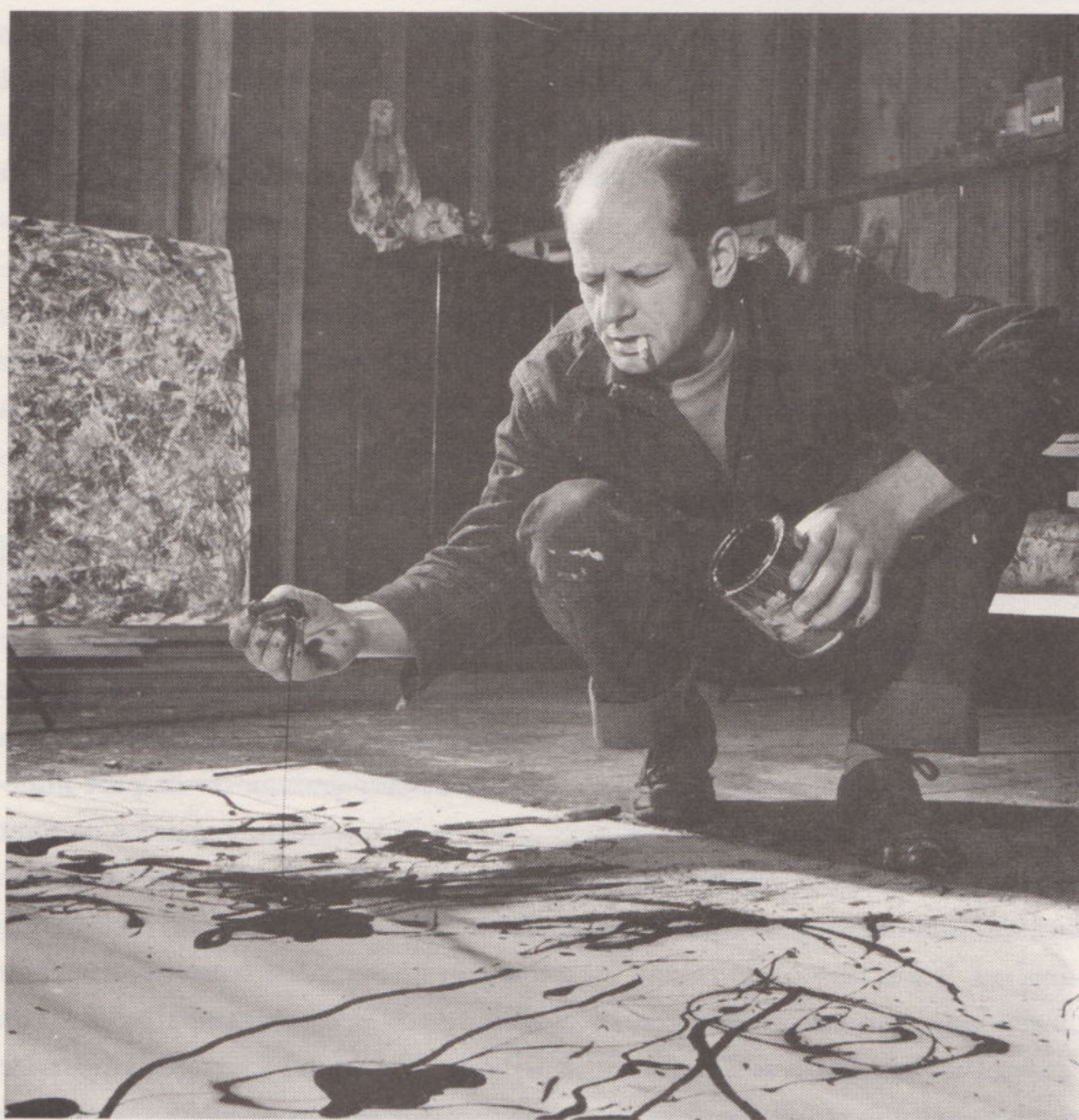
Here I love you.
 In the dark pines the wind untangles itself,
 the Moon phosphorescent on the errant waters.
 The days, each equal to the next, chasing one another.
 The fog unfurls into dancing figures.
 A silver seagull lowers from the Sunset.
 Sometimes a sail. High, high stars.

In those seven lines, the poet's perspective has brought together three misleadingly three-dimensional objects; think of the dense mesh of pine needles, where the wind untangles itself; of the porous foam that tops off those errant waters on which the Moon is phosphorescent, or the illusive breath of a dancing wisp of fog. To which we should add the ubiquity of the stars, those dots scattered in the sky forming a mosaic of light, the complexity of which our eyes cannot grasp, is almost two-dimensional. This ambiguity is, in fact, a consequence of the fractal nature of these objects. The problem is that our poor visual perception is incapable of appreciating the reality of its fractional dimension; a reality that, on the other hand, becomes clear when applying the sharp scalpel of Hausdorff's dimension or Neruda's acute dreamlike intuition.

The fractal nature of Pollock's drip and splash

Of the many applications of fractals that could be demonstrated, I am going to cover one that is highly unusual and one which links them with Jackson Pollock's abstract expressionism. I am referring to, of course, the American painter.

Pollock was the stereotypical troubled artist with problems with alcohol and other such things. He died in 1956 in a car accident at just 44 years of age. Peggy Guggenheim was his patron. Pollock once said, "The modern painter cannot express this age, the airplane, the atom bomb, the radio, in the old forms of the Renaissance or of any other past culture. Each age finds its own technique." True to this maxim, in the mid-1940s he created abstract expressionism. For this he



Jackson Pollock working in his studio in the late 1940s.

used large canvasses to which he applied his drip and splash technique. In 1946 he converted a farm in Long Island into his studio; placing the canvasses on the floor. "On the floor I am more at ease," stated the artist, "I feel nearer, more part of the painting, since this way I can walk around it, work from the four sides and literally be in the painting. I continue to get further away from the usual painter's tools, such as easel, palette, brushes, etc. I prefer sticks, trowels, knives and dripping fluid paint or a heavy impasto with sand, broken glass or other foreign matter added." One critic said: "The painting is not Art; it's an Is. It's not a picture of a thing. It's the thing itself.... it doesn't reproduce Nature; it is Nature." And it is true because his paintings ooze movement, graphical rhythm, vitality – not without violence – and at the same time contain a deep sense of tenderness.

The relationship between Pollock's paintings and fractals was shown by three Australian scientists: R. Taylor, A. Micolich and D. Jonas. In 1999 they published an article in *Nature* in which they announced that Pollock's paintings from the drip and splash period had fractal structures, generated both by how the painting dried (differences in the widths of the drops and trails) and by the geometric configuration followed by the trails which the painter scattered during his trips around the painting. The scientists even measured the fractal dimension of these structures (using the procedure explained above). The Australians' calculations showed that this dimension began to have values of greater than 1 (in other words, his paintings began to be truly fractal) in the mid-1940s; from that time it constantly and progressively increased until reaching values in 1952 of close to 1.7 for the dimension of the chaotic patterns generated by the scattering of paint, and 1.9 for the dimension of the chaotic configurations owed to Pollock's movement. The pattern of growth in those numbers was so uniform, and such was its regularity in the work analysed, that it could be used to determine the authenticity of Pollock's work, and even to date it.

Of course, Pollock did not purposely control these fractal dimensions. He was undoubtedly not aware of the fractal character of his paintings. It was pure intuition, pure style, pure skill. Pollock's way of working is well known; there are films showing him in action. He repeatedly retouched and tinkered; and added drops here and trails there, in a process that could take months, before declaring the painting finished. It was very complex and we know that he discarded many canvasses that he felt unhappy with – and that he cut the borders off others because the outer area was not as convoluted as the rest of the painting. The fractal dimensions that were so characteristic, which the Australian scientists managed to calculate, are nothing less than a measurement of his style.

Now let's return to the query with which this chapter began. For all the aesthetic harmony and value that a fractal has, it is possible that our increasingly put-upon passer-by may continue to think that fractals say little about the human condition.

Our imaginary vox pop might say that those Apollonian packings or the Koch curve are certainly beautiful, and that the Hausdorff is a perfect measure of their strange beauty. But also, surely our hypothetical interviewee would add that this theory, based on counting small squares and then calculating with logarithms, appears just like the typical strange digression of a few pointless mathematicians who are far removed from the world that surrounds them. And they would be very wrong because nobody can be removed from the world that surrounds them. Not even the actual concept of the Hausdorff dimension can, as I explained previously, because that concept is the work of humans it also has emotional context – those of its discoverer and the world in which he lived. Hausdorff is not just an empty name, or a hollow label, but a being of flesh and blood with feelings, dreams, problems and everything else. In his day, he sailed the river of life just like the readers of this book and I are doing now.

The passer-by might say, “What is a science that generates concepts with the level of abstraction of Hausdorff's dimension going to teach us?” You may make your own judgement from what follows on whether the contraposition of this mathematical concept with its emotional circumstances sheds a little light on the unfathomable darkness of human nature.

Hausdorff: the most Borgean of mathematicians

There are, perhaps, no better adjectives to qualify most of Felix Hausdorff's mathematical output than those that are often applied to Argentinian writer Jorge Luis Borges' works of fiction: “imaginary”, “paradoxical”, “ironic”, “labyrinthine”.

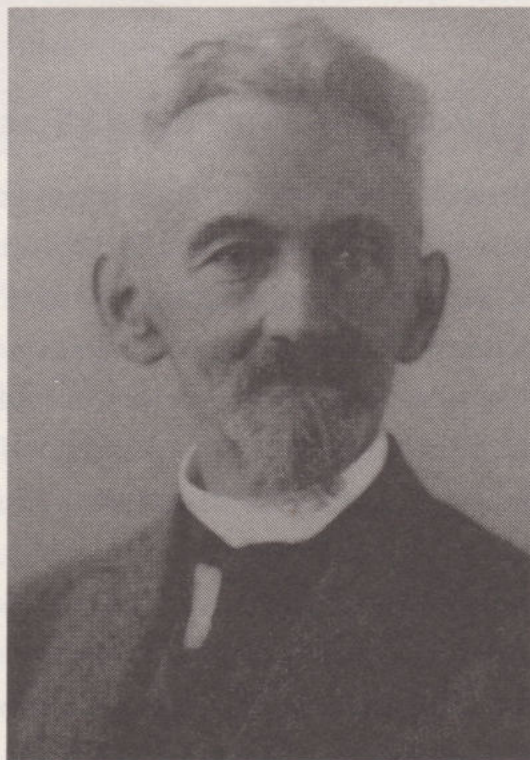
The Hausdorffian concept of dimension enriched the classic concept and allowed a better classification of objects. Thus, fractals, those labyrinthine objects par excellence, which Benoît Mandelbrot made so famous and popular in the final quarter of the 20th century, are described precisely as sets with a Hausdorff dimension that is not a natural number.

Hausdorff also considered the precedent of what we today call ‘inaccessible cardinals’. These infinite sets are guiding principles that have an unequivocal sense of irony. The characteristic that determines them is their colossal immensity; but this amorphous giantism makes them so improbable that we are ignorant that they

really exist. Here lies their irony. They are so large in size, nobody would have thought that our mind's eye would have had such difficulty in seeing them! Because the unreality of inaccessible cardinals persists even when the most vaporous and imprecise interpretation that we mathematicians have been able to imagine is applied to the term *existence*.

And the labyrinthine and ironic are not the only things to be found in Hausdorff's mathematics, the contradictory also plays a leading role. Undoubtedly, the Hausdorffian peak of contradiction is the description he made in his book *Basics of Set Theory* about the paradoxical decomposition of a spherical surface – the origin of the deconstruction of a solid sphere that would be carried out ten years later by the Poles Banach and Tarski. This would allow it to be divided into pieces (five, for example) and obtain from them two spheres that are identical to the original one. In addition it could divide a pea into conveniently designed pieces, such that, if reorganized appropriately we can obtain a solid sphere the size of the Sun. It is the mathematical version of the evangelical multiplication of the loaves and fishes.

Hausdorff was born in Breslau in 1868, although three years later he and his family moved to Leipzig. There, and later in Freiburg and Berlin, he studied mathematics and astronomy. Despite the fact that the mathematical work of his youth falls into the category of what is understood to be applied mathematics



Another photograph of Felix Hausdorff. The mathematician's expression transmits a melancholic light. Might he already see before him the storms which, according to Nietzsche, shake the trees of life?

(astronomy and optics, for example), Hausdorff ending up being a pure mathematician. His most influential work could be *Basics of Set Theory*. This monumental book, published in 1914, is considered to be the founding act of topology.

Hausdorff had other intellectual concerns besides mathematics. From adolescence he wanted to study music and become a composer and, although after his professional career he explored different paths, he composed a few pieces and was always a consummate pianist.

Under the pseudonym of Paul Mongré, Hausdorff wrote poetry, philosophical essays and also a satirical play. His literary output was mainly centred between 1896 and 1906. In 1897 he published his first book, *Sant' Ilario: Thoughts from Zarathrustra's Country*, the title of which is a reference to a coincidence. He started to write it on the day of Sant' Ilario in Sant' Ilario, a small village near Genoa. In this text we can find fairly philosophical sonnets, poems and aphorisms; one of them states: "When we have no woman to love, we love humanity, science or eternity... Idealism is always a lack of something better, a substitute for erotica."

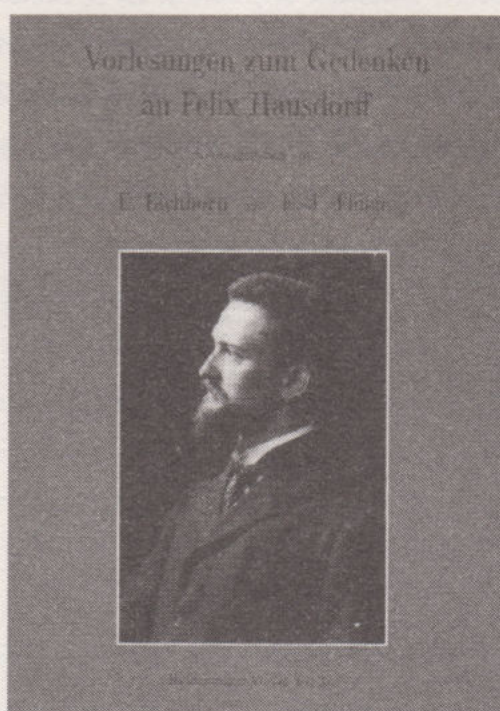
In 1898 *Chaos in Cosmic Selection* appeared. In it Hausdorff champions 'transcendental nihilism'. It was said of the book that it was too mathematical for a philosopher and too philosophical for a mathematician. Hausdorff's philosophy was heavily influenced by Nietzsche and Schopenhauer, and postulated the advantage of elitist over egalitarian societies. Hausdorff used to break up the intelligent philosophical discourse of his books with reflections that were, let's say, less lofty, in relation to egoism, hedonism, love, passion, the music of Mozart or hypnosis (Freud's influence can clearly be seen in his writings).

Apart from the sonnets contained in *Sant' Ilario*, Hausdorff published another book of poems titled *Ekstases* (1900), with 158 compositions. So that you can appreciate Hausdorff's poetry, here is one of his poems called *Unendliche Melodie* (Endless Melody):

Slowly set off on courses bent
 where the iron sounds of the beginning hark,
 into smoke and world in spiral dance dark
 the soul evolves in the firmament.
 Sight without hindrance or impediment
 by an angle or corner or mark,
 slowly set off on courses bent
 where the iron sounds of the beginning hark,

From all singularity exempt,
 unrelated to man, pure lark
 a sound without restoring spark,
 to float, to pass by formless, movement,
 slowly set off on courses bent.

However, it was his play that achieved the most success. It shares its title with a drama by the Spaniard Calderón, *The Doctor's Honour*, although Hausdorff's approach is quite a lot more satirical and direct. The play tells the story of a Prussian architect, an idealist who, having seduced the wife of a state councillor, has to fight a duel with him. But, when the agreed date and time arrive, the clash has to be suspended due to both contenders, and their respective witnesses' alarming state of drunkenness. As a result of the scandal, the councillor loses his job but gets his wife back. The play was performed in Berlin and Hamburg and, according to the local press, was warmly received.



Hausdorff's achievements have been the subject of countless analyses, such as that collected in this joint project run by E.J. Thiele and E. Eichhorn.

"Withered crowns on the sanctuary of life"

Hausdorff was a professor at the universities of Leipzig (1902–1910), Greifswald (1913–1921) and Bonn (1910–1913 and 1921–1935). He retired from the last post in March 1935, when he was 67 years old and, as he had stated years before, things

in Germany were starting to change. I have spoken of Hausdorff the mathematician, the writer, the philosopher and the musician, but now is the time to talk of him as the German patriot which he always considered himself.

On 30 January 1933 Paul von Hindenburg, the then president of Germany, had appointed Adolf Hitler as chancellor. The Nazi Party, with 230 elected members of parliament in 1932, had majority representation in the House, although its majority was not absolute. In the first elections of March 1933, the Nazis obtained 288 members of parliament, which, together with the 52 from the National Party, gave Hitler effective control of a parliament with 647 deputies. The subsequent exclusion or arrest of the 81 communist members of parliament, and a deal with the Centre Party, allowed Hitler to obtain the two-thirds of the votes that he needed for the proclamation of the Third Reich and the assumption of absolute power.

True to his anti-Semitic discourse, Hitler did not take long to approve the first laws of ethnic exclusion. On 1 April 1933, he called for a boycott against Jewish businesses. A week later, on 7 April, a law was decreed reforming the public administration, which impeded Jews from working for the state – those who had done so until then were fired.

At the time, there were 200 mathematics professors (of whom 98 were heads of departments) in German universities; 35 were dismissed, 15 of whom were heads of departments. Of those dismissed, 30 were deemed to be Jewish “to a certain extent”. By 1935, the figure was around 60. Sixty is just a number: behind it are 60 tragedies, 60 infamies, 60 people, 60 families, many of whom were eventually exterminated.

The law of 7 April, however, had some exemption clauses: any Jews who had shown themselves to be German patriots were exempt (for example, this applied to those had fought in World War I), and could continue to be civil servants.

This was the case for Hausdorff. He never hid his Jewish origins, though religious issues are not abundant in his writing; where he does touch on them there are far more pages on oriental religions than on Judaism or Christianity. His wife, Charlotte Goldschmidt, whom he married in 1899 and with whom he had a daughter, Lenore, had converted to Lutherism in her youth.

Despite the fact that Hausdorff soon felt the fury of the Nazi monster (after Hitler's party obtained 230 members of parliament, he had said: “Things will be very different in the future...”), he made no determined attempts to leave Germany. We only know of a letter sent to Richard Courant (1888–1972) in 1939 informing him of a possibility of a position as a researcher in New York.

Possibly, if Hausdorff had been dismissed from the University of Bonn, things would have been different for him and his wife. But Hausdorff was considered a patriot who, in his youth, just after graduating, had served several years as a volunteer in the German infantry, reaching the rank of vice-sergeant. So the exemption to the law of 7 April applied to him and he continued to be a head of department at Bonn until his retirement in March 1935.

COURANT, HILBERT AND GÖTTINGEN

Richard Courant, a German Jew like Hausdorff, quit Germany in 1933. Despite having been injured during World War I, despite the many characters who sent letters from around the world to the University of Göttingen requesting that he keep his position, and despite being an excellent scientist, the University dismissed him on 13 April, 1933, alleging that Courant was a member of the Socialist Party. Paradoxically, this outrage ended up saving his life. In 1936 Courant managed to get a place at the University of New York. There he ran the Institute of Applied Mathematics (one of the most prestigious in the world) which has carried his name since 1964. Courant, of course, was not a one-off at Göttingen.

From Göttingen, the Third Reich's ethnic policies had cut off figures of such stature as Courant, Edmund Landau, Emmy Noether and Hermann Weyl (and many more). Many of them belonged to the school of David Hilbert, who had not allowed any prejudice, be it nationalistic, racial or sexual, to affect him when it came to selecting students and collaborators, and who with great effort and endeavour managed to keep Göttingen at the mathematical centre of the world. In just a few years, Göttingen became practically insignificant. "When I was young,"

Hilbert commented at the age of 71, "I decided I would never repeat what I had heard from so many older people: 'They were good times, not like today.' I decided I would never, ever say that when I was old. But, now, I have no choice but to say it."



David Hilbert remained at the University of Göttingen despite the "purge of 1933", which led to the expulsion of numerous researchers. The illustrious mathematician died ten years later.

However, his ordeal still lay ahead of him. In April 1941, a colleague of Hausdorff's wrote of him and his wife: "Things are going tolerably well for the Hausdorffs, although of course they cannot escape the harassment and agitation caused by the continuous anti-Semitic legalisms. The fiscal and monetary duties which have been imposed on them are so high that they cannot live on their retirement salary and they have had to dip into their savings, which they fortunately still had. They have also been obliged to yield part of their house and now too many people live there [...] It is certainly encouraging that musicians still visit to play with Hausdorff: at least it brings a little joy to the house."

In October 1941, the Hausdorffs were forced to wear the Star of David, and towards the end of the year they received the news that they would be deported to Cologne. This was the first step before internment in the concentration camps that Hitler had established in Poland. The threat appeared to vanish at New Year, only to give way to another one. In the middle of January they were informed that on the 29th of that month they would be interned in the suburb of Bonn called Endenich. It was, again, another step on the way to a death camp.

A letter that Hausdorff wrote on Sunday 25 January 1942 has survived. In it he stated: "*Auch Endenich ist noch vielleicht das Ende nicht.*" The sentence is a gruesome play on words, using Endenich, a Bonn neighbourhood, and *ende* and *nicht* which mean 'end' and 'no': "Although Endenich may still not be the end." As Hausdorff was an amateur musician, he would undoubtedly have known that there was an asylum run by a Doctor Richarz (which perhaps no longer existed in 1942); a gloomy place where the composer Robert Schumann (1810–1856) spent the last two years of his life shut away. Certainly a bad omen.

So "Although Endenich may still not be the end" is a pun. And surely one of the most darkly ironic puns that has ever been written, because the Hausdorffs had decided to commit suicide. "For when you receive these lines," reads the letter, "we will have resolved our problem; although it will be in the way from which you, tirelessly, have tried to dissuade us [...] What has been done to the Jews in recent months has plunged us into the most absolute grief, because it has brought us to an intolerable conjuncture [...] Please give Mr Mayer the most heartfelt thanks for everything he did for us, but also for everything which he certainly would have done; we sincerely marvel at the achievements and success of his organisation and, had we not fallen into this sorrow, we would have welcomed his concerns. Perhaps they would have procured us a sense of relative security, although unfortunately it would not have ceased to be relative." Hausdorff was right. Mr Mayer, a lawyer,

died in Auschwitz. The letter continues: “[...] If possible, we would like our bodies to be cremated; I attach three declarations to this end. If it is not possible, may Mr Mayer or Mr Goldschmidt do whatever they can (take into account that my wife and sister-in-law are Lutherans). I assure you that whatever the cost, it will be paid: my wife has already paid the expenses of her burial at a protestant foundation (you will find the documents in her bedroom). Anything that remains to be paid, will be paid by my daughter Nora. Forgive us for causing you problems, even after our deaths. I am convinced that you will do what you can, which perhaps will not be much. Excuse our desertion! We wish you and all our friends a better future.”

Hausdorff showed in that letter, written hours before committing suicide, a truly surprising presence of mind. Hausdorff had written of suicide every now and then, and these reflections perhaps helped him to face his fate, although who can say what will, or will not, be of aid when their time comes. Hausdorff had published an essay in 1899 entitled *Death and Return*, heavily influenced by Nietzschean thinking on ‘the free life’. In that goodbye letter written by Hausdorff on the morning of his death, the ideas of Zarathustra still resound strongly. “Die in time!” Hausdorff’s seems to scream out to us, as if he wanted to show us by his dignified conduct that “that which is completed dies a death victoriously”. Hausdorff was no longer prepared



The grave of Felix Hausdorff, his wife Charlotte and his sister-in-law Edith in the Poppelsdorf cemetery in Bonn.

to “hang withered crowns on the sanctuary of life”, so he chose “free death, which comes to me because I want it to”.

On the afternoon of the day he wrote that letter, Hausdorff, his wife Charlotte and her sister, Edith, took an overdose of veronal. It seems that it was possible to comply with his wishes, as their remains were cremated and the ashes, deposited in the Poppelsdorf cemetery.¹

Coda

Perhaps the hypothetical passer-by, who was questioned at the beginning of the chapter about mathematics and the human condition, would tell me that what I have said so far does not do much to help understand us as a species. And that the juxtaposition of the abstract of the Hausdorff dimension and the emotional context of his life, rather than enlightening the human condition, throws up a good number of questions and conundrums. Even so, this juxtaposition could be of great benefit, because it is precisely the reflection that the questions and conundrums provoke that fuels the mind to illuminate the starkness of our condition.

¹ The emotional upheaval experienced by mathematicians in the tragic years leading up to, and during, World War II go far beyond the description here. We should not forget that the Nazi barbarity ended up crossing Germany's borders, plunging the rest of Europe into horror. Among many other things, we see the annihilation of Polish mathematicians and, as these horrors led to the diaspora of many European scientists and mathematicians to America, involvement in the design and manufacture of the first atomic bombs – one of the most dramatic and morally complex episodes which science and technology have ever had to face.

Chapter 4

Objective: The Beauty of Mathematical Reasoning

We now know where to look for the beauty in mathematics and why it is difficult to appreciate. We also know about the importance of taking the emotional circumstances of mathematics into account, as well as the techniques, because it sets the stage for appreciating its beauty.

In this chapter we are going to put ourselves on the other side of the mirror. Let's place ourselves in a situation where the structure of ideas where aesthetic value resides in a particular piece of mathematical reasoning has already reached our brain. What we are going to attempt now is a kind of double leap: I will try and make explicit which characteristics of these ideas are those that give them their aesthetic value. But do not worry, I will always bear in mind the philosopher George Santayana's recommendation: "To feel beauty is better than understanding how we come to feel it."

I am going to make this trip arm in arm with a giant (people do not stand on their shoulders any more) who will advise us and accompany us on this troublesome journey. That giant is G. Harold Hardy (1877–1947): pacifist in times of war, celibate homosexual (in the words of his closest collaborator), stylist of mathematics, gourmet chef of numbers.

The Englishman who did not love God

In harmony with the rest of the book, before moving on to Hardy's arguments on the aesthetic value of mathematics, it is worth getting to know some of our character's back story.

Harold Hardy was – in his own estimation – the fifth best pure mathematician of his time. He was particularly fond of establishing rankings (which may reveal nothing more than his competitive character). On another occasion, Hardy created a ranking on natural mathematical ability, in which he awarded himself

25 out of 100, while giving his closest collaborator, John Littlewood, 30, and the German David Hilbert, the most renowned mathematician of the time, 80. The maximum score of 100 was given to Srinivasa Ramanujan, a self-taught Indian mathematician, a former office worker at the port of Madras, a rough diamond whom Hardy had the professional honour of discovering (and whom we will have more time to discuss later).

In Bertrand Russell's opinion, Hardy had the type of bright eyes that only very intelligent people have. Perhaps he was not a genius like Einstein but, for many, he had a facet in which he surpassed him: an ability to convert any work of intellect into a work of art – a capability which he made patent in a booklet that he wrote a few years before his death. It is called *A Mathematician's Apology*, and according to Graham Greene, it is the best description of what it means to be a creative artist. This pamphlet will be the compass that guides us in our task of objectifying the characteristics which give a mathematical idea aesthetic value.



Someone once said that in order to sit as Hardy is in this photograph you must have been educated in a British public school.

Hardy received an excellent English education, first at the school in Surrey, where his parents worked as teachers; then, at the age of 13, he obtained the first place out of 102 candidates to study at Winchester, a prestigious private school. And, finally, at the age of 19, at Trinity College, Cambridge.

Hardy was reserved but also had a somewhat histrionic and eccentric personality. His phobia of mirrors (when he arrived at a hotel he would cover them with towels) forced him to shave by touch. This neurosis extended to some mechanical devices – he never used a watch or a fountain pen, and he abhorred having his photo taken. He did not use the telephone either, except in extreme situations in which only he would speak. Apart from mathematics, his other great love was cricket, a game that is almost as arcane in its complexities as mathematics itself.

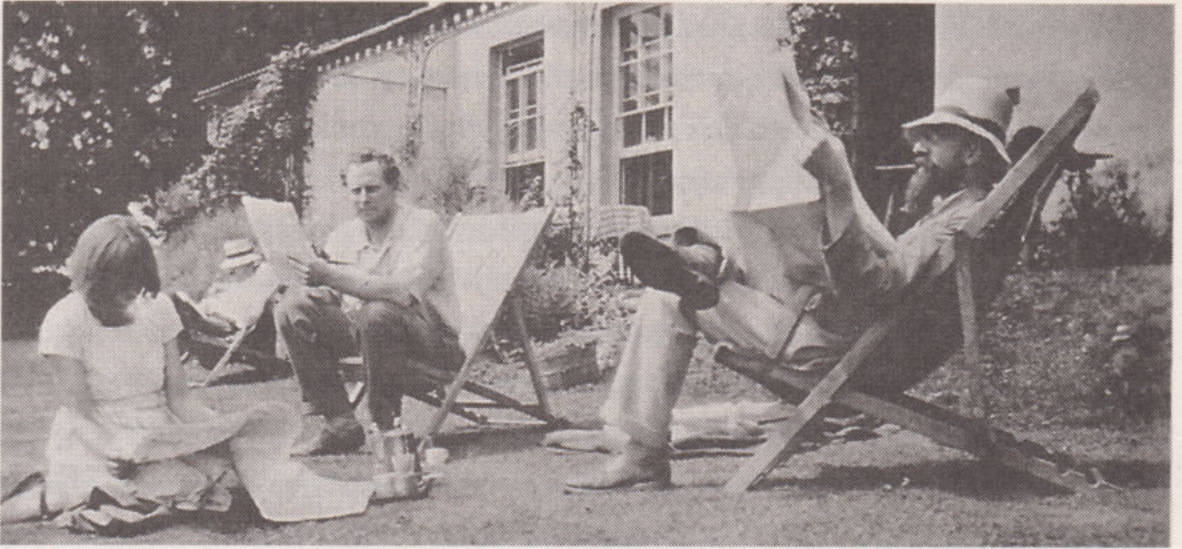
A good friend of Bertrand Russell and follower of his pacifist stance in World War I, Hardy was a loyal defender of integrity and the universality of the mathematical brotherhood. The atmosphere at Cambridge during World War I led him to change university, accepting a chair at Oxford in 1919. He returned to Cambridge 12 years later. He missed being at the centre of English mathematics – which was Cambridge in those days – and was also offered the use of college accommodation even after he had retired. Hardy was always single, and he was cared for in his final years by his sister Gertrude, who had a glass eye because her brother hit her with a cricket bat when they were playing as children. (The accident never affected the excellent relationship that the siblings maintained throughout their lives.)

Hardy was very conservative, although perhaps his apparent sincerity and grace hid more than it revealed. As a consequence, it has been said, he was a friend of many but intimate with very few.

Apostles

Hardy belonged to the exclusive Apostles club, a strange secret brotherhood created at Cambridge which had intellectual members of great pedigree and prestige: Edward M. Foster, John Maynard Keynes, Bertrand Russell, Ludwig Wittgenstein, Lytton Strachey and other members of what would later be called the Bloomsbury group. “There were no taboos,” wrote Russell of the Apostles “no limitations, nothing considered shocking, no barriers to absolute freedom of speculation.”

Despite the fact that many of the Apostles were not homosexual, there was a camp streak running through the brotherhood. Foster, Keynes and Strachey were homosexual and possibly Hardy too, although his discretion makes it difficult to be sure. According to Littlewood, with whom he collaborated for 40 years: “Hardy was a non-practising homosexual.” The closest thing to an affair he ever knew was in 1903 when he shared a room and a cat with Russell Gaye, an expert in the Greek and Roman classics. “They were absolutely inseparable,” wrote the husband of Virginia



Three members of the Bloomsbury Group, some of whom Hardy was associated with. On the right is writer Lytton Strachey; on the left, painter Dora Carrington, and in the middle, her husband Ralph Partridge. The relationships between these three characters were presented in the 1995 film Carrington starring Emma Thompson.

Woolf. "They were never seen apart and rarely talked to other people." Six years later Gaye committed suicide. "One cannot conclude whether Hardy was a practising homosexual," wrote Robert Kanigel (in the biography of Ramanujan). "He may equally have had an asexual life or a secret intense sex life so well hidden that even his friends knew nothing of it and those who did said nothing." The only record we have of Hardy being sexually active refers to an Apostles meeting in 1899, when after a discussion about masturbation a vote was held as to whether it was a good means to a bad end, to which Hardy voted yes.

Hardy stopped believing in God while he was a student at Trinity College, Cambridge. Being true to himself, he told the Dean that he would not go to the chapel again, which was compulsory at the college; to which the Dean responded that he would not oppose this if, in exchange, he informed his parents. The Dean and, of course, Hardy knew that they were very religious, and that their son's atheism would cause them pain. After a lot of thought, Hardy wrote to them and, indeed, it caused them great pain. As a good friend of his said, a pain which, more than a century later, is difficult for us to imagine. So Hardy not only stopped believing in God, but from then on considered Him his personal enemy. He would sometimes go out well wrapped up and with an umbrella on sunny afternoons. Hardy was not overly cautious, because he thought that if God saw him equipped with an umbrella he would not bother ruining the afternoon with rain.

HARDY, GOD AND THE RIEMANN HYPOTHESIS

The most divine of the anecdotes regarding Hardy's enmity with God is related to the Riemann hypothesis. There is no need to explain this hypothesis here, you just need to note that, if it is correct, it would allow us to have very in-depth knowledge about how the prime numbers are distributed among the other numbers. Since it was proposed in 1859 by German mathematician Bernhard Riemann (1826-1866), the hypothesis has become the most important challenge in mathematics, and it was the object of Hardy's most intense mathematical desires. Just before embarking on a trip to Denmark, Hardy sent a postcard in which he claimed to have demonstrated the Riemann hypothesis. Should he disappear in a hypothetical wreck of the ship, his prestige would convince his colleagues that indeed he had resolved the most important problem in mathematics; they would think that only bad luck had stopped him from publishing a demonstration that had undoubtedly sunk with him. Thus, Hardy would enter into the mathematical Olympiad together with Gauss, Archimedes, Newton and Euler. Actually, what Hardy did was nothing more than imitate Pierre de Fermat and his famous note in the margin of Diophantus' *Arithmetica*. Only the ship did not sink. And Hardy later explained that it had all been a ruse so that he would have a peaceful trip – so that Hardy did not enter into the Pantheon of mathematical fame, God, his arch enemy, calmed the winds and the waters of the North Sea, and presented him with the calmest of journeys.

The relationship with Ramanujan

His relationship with Indian mathematician Srinivasa Ramanujan leaves Hardy's moral qualities in little doubt. Ramanujan was born in 1887 in a village to the south of Madras in a family of poor Brahmins. He was unable even to obtain the equivalent of a bachelor's degree, and not so much for economic reasons as because he was never interested in studying anything other than mathematics. He was a special boy who fell under the spell of numbers and, out of nothing, managed to construct a superb edifice of mathematics. And he did it on his own, without help from anyone, sitting in the doorway of his house, writing on a blackboard and erasing with his elbow.

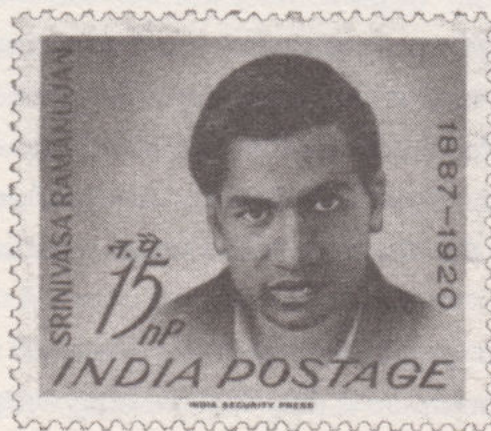
When word began to spread of his formulae, the small scientific community of Madras was unable to determine if they were dealing with a genius or a madman. Realising that nobody in his immediate surroundings could understand his formulae, Ramanujan decided to send them to Cambridge, the centre of Anglophone

mathematics at the time. But neither the first or second letters he sent had any effect, because the English professors to whom they were addressed to were too busy to try to understand what the unknown office worker from a distant land was sending them. The third letter fell into better hands – those of Harold Hardy.

Hardy took Ramanujan's letter seriously and, after studying what the Indian office worker had sent him in depth, he moved heaven and earth in order to bring Ramanujan to Cambridge. He got there in 1914, almost coinciding with the start of the World War I. Hardy was able to verify that Ramanujan was a rough diamond. He had a monstrously refined intuition for numbers and formulae, but was unaware of basic concepts and tools that any wannabe mathematician should master in the first years of their career. But the miracle appeared, the impossible happened, because Ramanujan, who had taught himself mathematics on the porch of his home in the south of India, was able to collaborate fruitfully and on an equal footing with Hardy, one of the most accomplished products of the English education system.

Ramanujan was in England for nearly five years (during which time the war came and went), although he spent the last two very ill and was admitted to several sanatoriums. The loneliness, the wet British climate, and the rigours of an inflexible diet (Ramanujan was a strict vegetarian) aggravated by the lack of appropriate foods during the war, made him sick with something that none of the several doctors who saw him were able to effectively diagnose.

Ramanujan returned to India in 1919, but he did so to die. He left his country chubby and healthy-looking and returned consumed by illness. He also left as an unknown and he returned as a celebrity, chosen as a member of the Royal Society of London, the youngest member in its one hundred year history and the second Indian, and he was the first Indian to be named a fellow of Trinity College, Cambridge. The



An Indian stamp dedicated to the mathematician Srinivasa Ramanujan, Hardy's great discovery.

Madras Times dedicated an article to him, where it could be read that: “As someone from Cambridge said, since Newton there has been nobody like him, which, it goes without saying, is the highest of praise.” Ramanujan died in April 1920 at the age of 32.

“My association with Ramanujan,” Hardy once confessed, “was the single romantic incident in my life. I owe more to him than to anyone else in the world with one exception.” Hardy was always proud of having collaborated with Ramanujan and he considered it a measure of his personal value. In *A Mathematician’s Apology* he wrote: “I still say to myself when I am depressed and find myself forced to listen to pompous and tiresome people, ‘Well, I have done one thing you could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms.’”

And that relationship started when, at the end of January 1913, with the morning post, he received an envelope sent from Madras and covered in Indian stamps. “Certainly, the most extraordinary letter I have ever received,” Hardy would later say.

Hardy lived almost exclusively by and for mathematics, and led English mathematics from the first decade in the 20th century until the start of World War II. To him mathematics involved a lot of competition – being the first in history to resolve a problem of recognised difficulty. Author of more than 300 research articles and 11 books, his scientific output was very extensive and covered nearly every branch of mathematical analysis and the theory of numbers.

Art and mathematics: purposefulness without purpose?

Hardy’s mathematical experience was mostly aesthetic. We can find continuous references to it in *A Mathematician’s Apology*: “Beauty is the first test. There is no permanent place in the world for ugly mathematics.”

He even contends that beauty is the only thing that gives his mathematics value, which is paramount to saying that it gives sense to his whole life: “Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow. I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. [...] The case for my life, then, or for that of anyone else who has been a mathematician in the same sense that I have been one, is this – I have added something to knowledge and helped others to add more, and that these somethings have a value that differs in

degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.”

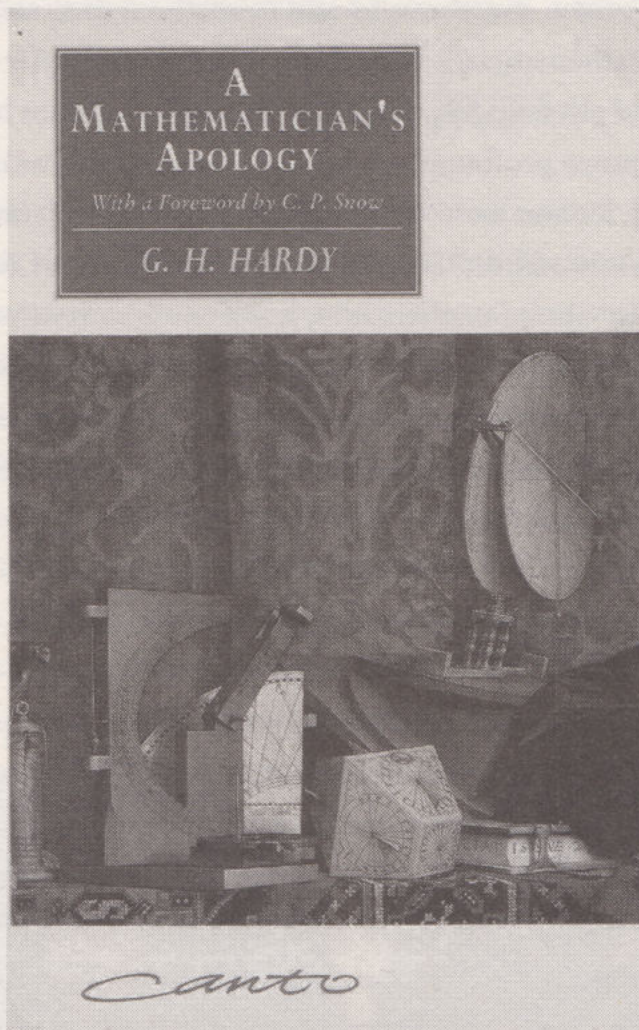
Often, Hardy’s insistence of the uselessness of real mathematics has been considered a demonstration of his most extravagant, and provocative, aspect. Because it is practically provocation to write: “Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.” This was said at about the time that the US was running the Manhattan Project and manufacturing the first atomic bombs. But, such are the ironies of life, the first thing the *Encyclopaedia Britannica* covers in its biography of Hardy is the so-called Hardy-Weinberg law, to which in a subsequent entry the *Britannica* dedicates more space than it does to Hardy himself, to say: “Hardy gave little value to this law, but its importance is central in the study of many genetic problems, including Rh distribution according to blood groups and haemolytic diseases.”

Although for me Hardy’s shameless praise of the uselessness of mathematics has another significance than that of being a mere feature of histrionics. Hardy, in his own way, was declaring that in matters of aesthetics he was a follower of Kant.

Aesthetic satisfactions seem to be of a different nature to the other satisfaction that has more to do with the biological state of our animal constitution, both in sensory and rational terms. Thus, the satisfaction felt by a prehistoric human with a decorated clay pot has little relation to that felt by satisfying their hunger or thirst from the same pot; and, in the same way, our delight at satisfying our sensual instincts seems to differ from that we get from listening to Rachmaninov’s *Piano Concerto no. 2*. According to Kant, the difference between aesthetic satisfaction and other satisfaction lies in the fact that the latter is produced by complying with some need and, therefore, it always has ulterior motives. Artistic satisfaction, on the other hand, has no type of ulterior motive. Man, Kant would come to say, is the only animal with the capacity for aesthetic judgement: “The ability to judge an object, or a way of presenting it, by means of liking or disliking devoid of all interest.” It is precisely this lack of interest that is the fundamental characteristic of a work of art. “Art,” Kant wrote in *Critique of Judgement*, “is purposefulness without purpose.”

So what Hardy sought by praising the uselessness of mathematics was in keeping with Kant’s theory of aesthetics – Hardy was championing mathematics as more art than science.

This revindication of aesthetics in mathematics, and therefore its uselessness, is present throughout *A Mathematician's Apology*. Given that it is the text I will use in what follows to make explicit which characteristics of mathematical ideas are those that give them aesthetic value, I should finish this section with a couple of paragraphs on Hardy's personal situation at its time of writing.



The front cover of A Mathematician's Apology.

Hardy's passion for mathematics eventually consumed him. At the end of his life, when his mental energy was not sufficient to produce mathematics, he became depressed and attempted suicide. It was precisely when he faced that last period of his life, then well into his seventh decade, that he wrote *A Mathematician's Apology*; and it is impregnated with this underlying bitterness. Charles P. Snow wrote in the prologue, "If read with the textual attention it deserves, this is a book of haunting sadness. Yes, it is witty and sharp with intellectual high spirits; yes, the crystalline

clarity and candour are still there; yes, it is the testament of a creative artist. But it is also, in an understated stoical fashion, a passionate lament for creative powers that used to be and that will never come again.”

Hardy said the same in the first lines of his book, but in the most brutal terms possible: “It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. Statesmen despise publicists, painters despise art critics, and physiologists, physicists, or mathematicians have similar feelings. There is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds.” And he continues: “If then I find myself writing, not mathematics, but ‘about’ mathematics, it is a confession of weakness, for which I may rightly be scorned or pitied by younger and more vigorous mathematicians. I write about mathematics because, like any other mathematician who has passed 60, I have no longer the freshness of mind, the energy, or the patience to carry on effectively with my proper job..”

Generality and depth

The purpose of this section is to provide the reader with a description of those characteristics that give mathematics aesthetic value. I will start by reminding you that, according to what Hardy himself wrote in *A Mathematician's Apology*: a mathematician is a builder of configurations whose basic material is ideas: “The mathematician's patterns, like the painter's or the poet's must be beautiful; ideas, like colours or words, must fit together in a harmonious way.”

In order to achieve our goal we must, therefore, identify which fundamental qualities give a mathematical idea its aesthetic value. Following Hardy, we will start by identifying two essential intrinsic elements (that is, aspects which qualify their own constitution as an object) capable of giving a mathematical idea aesthetic dimension. They are generality and depth.

An example from Euler as a starting point

We should start by illustrating Hardy's reflections on these characteristics. I have chosen an example that is not too complicated (so that a reader without too much

mathematical knowledge may follow it), but complex enough to allow me to adequately illustrate all Hardy's arguments regarding aesthetic value in mathematics and, also, allow the establishment of enriching relationships with other considerations on aesthetics borrowed from philosophy. The example I have chosen is the way in which Euler added the square of the inverses of the natural numbers in his book *Introductio in analysin infinitorum* (*An Introduction to the Analysis of the Infinite*). The sum in question that Euler managed to calculate is the following:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \dots$$

Note that the denominators are the squares of natural numbers, and that the ellipsis means that the number of addends is infinite. Mathematicians call an infinite sum a series; the value of an infinite sum is the number that we approach as more addends are added, such that the difference between that number and the successive sums disappears as more and more addends are added.

As with nearly everything in mathematics, the infinite sum above has circumstances which, after what was explained previously, I cannot deny the reader.

The history of the sum is as follows. In March 1672, a young twenty-something Leibniz arrived in Paris. Among his concerns was improving his mathematical education, which was at the time quite lacking. Not many months later he had



On the left, Leibniz, painted by Wentzel in around 1700; and on the right, Huygens, in a canvas painted by Netscher in 1671.

come up with an ingenious procedure for calculating infinite sums. It consisted of writing the addends as subtractions so that the successive cancelling of some terms with others allowed the final calculation of the sum. Leibniz's inherent optimism and mathematical amateurism at that time led him to think that he had found a method that allowed him to calculate any sum. Let's not forget that Leibniz stated that we live in the best of worlds possible, and for a German to say that not long after the end of the Thirty Years' War is something in itself. (Voltaire did not have to exaggerate his caricature too much when he decided to dress Leibniz in the clothes of Pangloss, the optimistic mentor of Candide.)

But this optimism of Leibniz's about his method for adding series was reinforced when he told Christiaan Huygens of his discovery. At the time Huygens was one of the most prestigious scientists in Europe. Born in the Netherlands, he had then been working at the Académie des Sciences in Paris for a few years. To test Leibniz's method, Huygens proposed adding the series of inverses of the triangular numbers. Triangular numbers have the form $n \cdot (n + 1)/2$, and their name comes from the Pythagorean geometric conception of numbers – with the units of a triangular number, equilateral triangular configurations can be formed. So the sum Leibniz had to calculate was: $1 + 1/3 + 1/6 + 1/10 + 1/15 + 1/21 + 1/28 + \dots$

As this series is, coincidentally, one of the few that can be added by applying Leibniz's procedure, he obtained the correct answer (see box opposite) by using it: $1 + 1/3 + 1/6 + 1/10 + 1/15 + 1/21 + 1/28 + \dots = 2$.

In 1673, Leibniz visited London and there he demonstrated his somewhat foolish optimism and his no less imprudent scientific dilettantism. Mathematically speaking, his attitude led him to put his foot in it on several occasions, which the English did not fail to remind him of 40 years later in the middle of the dispute with Newton over the invention of infinitesimal calculus.

On returning to Paris, Leibniz received a letter from John Collins in which he proposed adding the inverses of the squares of the natural numbers: $1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/36 + 1/49 + \dots$

Collins was not what you would call a great mathematician; his work was more that of an intermediary between British mathematicians and those on the continent. His mathematical ability would not have allowed him to understand the difficulty of the problem, so it is very possible that the proposal had been suggested by mathematicians of a much higher level (for example, James Gregory or, even, Isaac Newton). In any case, the problem's malicious proposer, whoever it was, could have told Leibniz that the calculation of that sum cannot have been too complicated as

SUBTRACT WHEN YOU WANT TO ADD

As stated previously, the Leibniz method consists in writing each addend in the sum which we want to carry out in the form of a subtraction, so that the successive cancellation of some terms with others would allow the sum to be calculated with ease. This is precisely what happens with the inverses of triangular numbers. Thus, the inverse of each triangular number $2/(n \cdot (n + 1))$ is the difference between $2/n$ and $2/(n + 1)$:

$$\frac{2}{n \cdot (n+1)} = \frac{2}{n} - \frac{2}{n+1}.$$

Now if we substitute values of 1, 2, 3, 4... for the variable n , we get: $1 = 2 - 1$; $1/3 = 1 - 2/3$; $1/6 = 2/3 - 2/4$; $1/10 = 2/4 - 2/5$; $1/15 = 2/5 - 2/6$; $1/21 = 2/6 - 2/7$... And so on. Now, adding the fractions written in this way, we will see that the subtrahend (the number being subtracted) in each difference is cancelled by the minuend (the number from which something is subtracted) such that all that is left at the end is the minuend from the first subtraction: $1 + 1/3 + 1/6 + 1/10 + 1/15 + 1/21 + 1/28 + \dots = 2$

the addends were almost equal to those of the series which Leibniz had already managed to add. In the first case those addends took the form of $2/(n \cdot (n + 1))$ and in the second, $1/(n \cdot n)$.

Leibniz was not able to add the series that they sent him from England, and nor could his disciples, the brothers Jacob and Johann Bernoulli. There is no documentary evidence that Gregory or Newton studied the problem, although this does not mean that they did not, probably with the same lack of success as the others.

Nearly half a century passed until someone would achieve it. That someone was Leonhard Euler.

The idea Euler used to add the series of the inverses of the squares of the natural numbers is very simple. The start point is the following: let's consider a product such as $(1 - 2z^2) \cdot (1 - 5z^2) \cdot (1 - 6z^2)$, and expand it into powers of z :

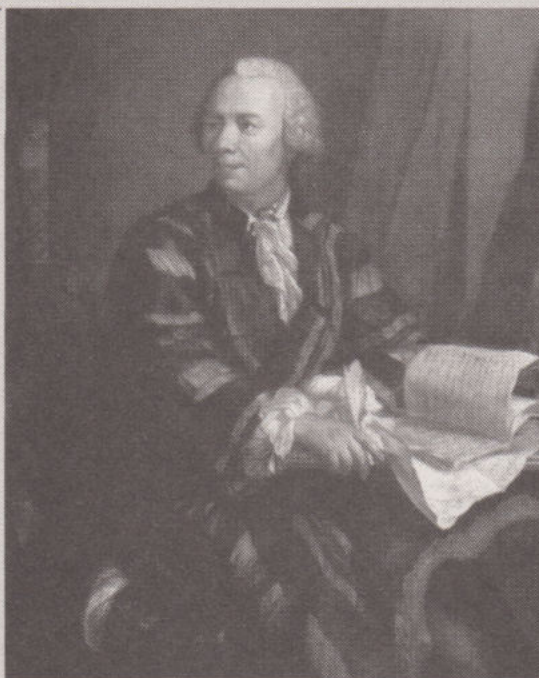
$$(1 - 2z^2) \cdot (1 - 5z^2) \cdot (1 - 6z^2) = 1 - 13z^2 + 52z^4 - 60z^6.$$

It is not difficult to see that the number that multiplies z^2 in the expanded expression is exactly the sum of the numbers that multiply z^2 in the unexpanded product on the left; and this is true whatever the number of factors that appear in this

LEONHARD EULER (1707–1783)

Euler is one of the greatest mathematicians of all time and, without a doubt, the most brilliant of the 18th century. Born in Basel in 1707, he studied at the local university and also received private classes from Johann Bernoulli (one of Leibniz' followers).

In 1727 he joined the St. Petersburg Academy of Sciences, where he worked until he moved to the Berlin Academy in 1741. Despite disputes with King Frederick II of Prussia, he endured Berlin for 25 years and ended up president of the academy. According to Frederick II, who made Berlin one of the most active cultural centres in Europe, Euler



lacked the brilliance, genius and elegance that was essential for the 'high life' so loved by the king. Instead, Euler was simple, direct and ingenuous, nothing like Voltaire (another of Frederick's recruits) who so delighted the king (before disgracing himself). In a letter to the French philosopher, Frederick II referred to Euler as "the great cyclops of geometry" – a joke in terrible taste as Euler had lost his sight in one eye in 1738. After his time in Berlin, Euler returned to the St. Petersburg Academy. He died there in 1783.

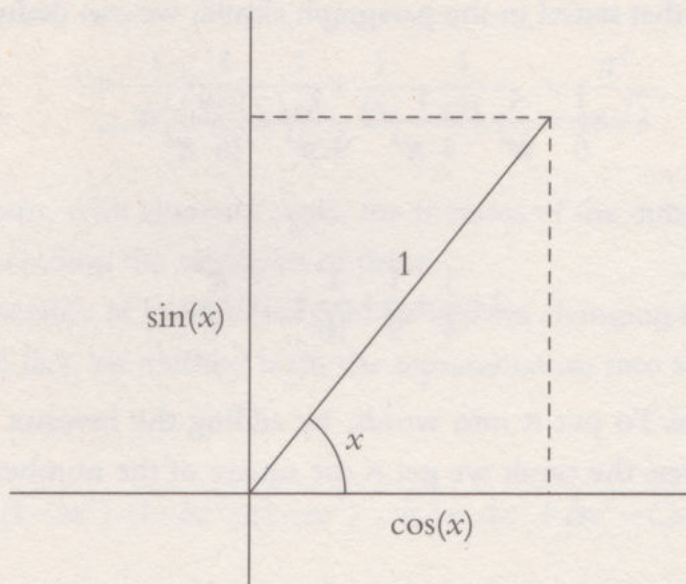
On his influence on later mathematicians I need only remind you of the classic Laplace quote: "Read Euler, read Euler, he is the master of us all!" Or the following from Gauss, which is lesser known but equally illustrative: "The study of Euler's work will remain the best school for the different fields of mathematics."

product, as can be easily proved. Euler understood that whatever was true of finite products and sums was also true for infinite products and sums; that is, if we write:

$$(1 - az^2) \cdot (1 - bz^2) \cdot (1 - cz^2) \cdot \dots = 1 - Az^2 + Bz^4 - Cz^6 + \dots$$

then $A = a + b + c + \dots$

Next, Euler brought the sine function into play. The sine function, together with the cosine, are the two basic trigonometric functions. Its definition is very simple. Using coordinate axes, we draw an angle x as follows: of the sides that form the angle, one is the horizontal axis, while the other's length equals 1. The sine is the length of the projection from that side on the vertical axis, while that of the cosine is the length of the projection on the horizontal axis, as shown in the figure.



Euler then considered two representations of the sine function; one was the infinite product representation, which he himself had discovered:

$$\frac{\sin z}{z} = \left(1 - \frac{z^2}{\pi^2}\right) \cdot \left(1 - \frac{z^2}{4 \cdot \pi^2}\right) \cdot \left(1 - \frac{z^2}{9 \cdot \pi^2}\right) \cdot \left(1 - \frac{z^2}{16 \cdot \pi^2}\right) \cdot \dots,$$

where in the denominator of z^2 the squares of the natural numbers multiplied by the square of the number π appear. And the other was the infinite sum representation, discovered by Newton:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \frac{z^6}{720} + \dots,$$

where the rule followed by the denominators is the factorial: the factorial of a generic number n is obtained by multiplying that number by all numbers that are smaller than it: $n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$. So, the rule followed by the denominators in the above formula is the factorial of the exponent of z plus 1. Thus, if the exponent of z is 2, then the denominator is the factorial of 3: $3 \cdot 2 \cdot 1 = 6$; if the exponent of z

is 4, then the denominator is the factorial of 5: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. And so on. As the two representations represent sine, both must be the same. Specifically:

$$\left(1 - \frac{z^2}{\pi^2}\right) \cdot \left(1 - \frac{z^2}{4 \cdot \pi^2}\right) \cdot \left(1 - \frac{z^2}{9 \cdot \pi^2}\right) \cdot \left(1 - \frac{z^2}{16 \cdot \pi^2}\right) \cdot \dots = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots$$

According to that stated in the paragraph above, we can deduce that:

$$\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{4 \cdot \pi^2} + \frac{1}{9 \cdot \pi^2} + \frac{1}{16 \cdot \pi^2} + \dots,$$

or:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}.$$

which is the same. To put it into words, by adding the inverses of the squares of the natural numbers the result we get is the square of the number π divided by 6.

Hardy's reflections applied

Now that we have seen the example of Euler, let's go back to Hardy's thematic thread of the essential properties whose presence gives mathematical ideas their aesthetic value. Hardy wrote: "There are two things at any rate that seem essential, a certain generality and a certain depth; but neither quality is easy to define at all precisely..."

Regarding the generality of a mathematical idea Hardy stressed: "A significant mathematical idea, a serious mathematical theorem, should be 'general' in some such sense as this. The idea should be one that is a constituent in many mathematical constructs, which is used in the proof of theorems of many different kinds. The theorem should be one which, even if stated originally in a quite special form, is capable of considerable extension and is typical of a whole class of theorems of its kind. The relations revealed by the proof should be such as to connect many different mathematical ideas." And so that there be no doubt about the difficulty of pinning down the term *generality*, he added: "All this is very vague, and subject to many reservations."

Let's illustrate with the example of Euler. Is there generality, in the sense explained by Hardy, in Euler's sum? There is – and in several senses.

Euler's fundamental idea is to use two expressions of a specific function, one in the form of a product and the other in the form of a series, to calculate the value of certain infinite sums. In the specific case above, using the sine function, Euler managed to add the inverses of the squares of the natural numbers. Using other functions, in the *Introductio* Euler managed to add a highly varied catalogue of infinite sums with the same method; for example:

$$1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \dots = \frac{\pi^3}{18\sqrt{3}},$$

where in the sum, with alternate signs, the inverses of the cubes of the odd numbers appear, excluding the multiples of three.

But the generality of Euler's idea goes far beyond changing the sine function for another one. In fact, his method takes the representation into account and

$$(1 - az^2) \cdot (1 - bz^2) \cdot (1 - cz^2) \cdot \dots = 1 - Az^2 + Bz^4 - Cz^6 + \dots$$

relates the number which multiplies z^2 in the developed expression with the sum of the numbers that multiply z^2 in the undeveloped product on the left. With a slight modification, Euler's idea is even more fruitful. All that we have to do is take a look at the numbers that multiply the other powers in the developed expression and write them in terms of the numbers that multiply z^2 in the undeveloped product on the left (see box overleaf). By developing this idea further, Euler managed to add not only the inverses of the squares of the natural numbers, but also the inverses of the fourth, sixth and eighth powers:

$$\begin{aligned} 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots &= \frac{\pi^4}{90}, \\ 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots &= \frac{\pi^6}{945}, \\ 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \dots &= \frac{\pi^8}{9,450}, \end{aligned}$$

and so, in a genuine orgy of infinite sums, he got to the power of 26:

$$1 + \frac{1}{2^{26}} + \frac{1}{3^{26}} + \frac{1}{4^{26}} + \dots = \frac{76,977,927 \cdot 2^{24}}{1 \cdot 2 \cdot 3 \cdot \dots \cdot 27} \pi^{26}.$$

I hope that this has allowed you to gauge the generality of Euler's reasoning, and by extension, better understand what Hardy meant when he talked of the generality of a mathematical idea: generality (as well as genius) is just what transcends this idea of Euler's.

EULER AND HIS ORGY OF INFINITE SUMS

Euler refined his idea as follows: let's reconsider the development

$$(1 - az^2) \cdot (1 - bz^2) \cdot (1 - cz^2) \cdot \dots = 1 - Az^2 + Bz^4 - Cz^6 + \dots$$

Now we are going to focus on the number B which multiplies the power of z^4 . It is not difficult to deduce that number B is obtained by multiplying two by two and adding the numbers $a, b, c, \dots$ which multiply z^2 in the undeveloped product on the left: $B = ab + ac + bc + \dots$

Thus, if we write: $P = a + b + c + \dots$ and $Q = a^2 + b^2 + c^2 + \dots$, a simple calculation tells us that: $P = A$ and $Q = A \cdot P - 2 \cdot B$.

If we reconsider the two developments of the sine function:

$$\left(1 - \frac{z^2}{\pi^2}\right) \cdot \left(1 - \frac{z^2}{4 \cdot \pi^2}\right) \cdot \left(1 - \frac{z^2}{9 \cdot \pi^2}\right) \cdot \left(1 - \frac{z^2}{16 \cdot \pi^2}\right) \cdot \dots = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots$$

and we take into account that in this case $A = 1/6$ and $B = 1/120$ and, according to the above calculation, $P = \pi^2/6$, we get the value of the sum of the inverses of the fourth powers of the natural numbers: $1 + 1/2^4 + 1/3^4 + 1/4^4 + \dots = \pi^4/90$.

We can do something similar with the power z^6 and the following powers, which allowed Euler to obtain the sums of the inverses of the even powers of the natural numbers, from the second until the twenty-sixth. A few years later, Euler found the general formula for the sums of the inverses of any even power of the natural numbers. Today we barely know anything about the value of the sum of the inverses of the odd powers of the natural numbers – just that the first handful of these numbers are irrational.

According to Hardy, the other intrinsic characteristic that gives a mathematical idea aesthetic value is depth. “The second quality that I demanded in a significant idea was depth, and this is still more difficult to define. It has something to do with difficulty; the ‘deeper’ ideas are usually the harder to grasp: but it is not at all the same. [...] It seems that mathematical ideas are arranged somehow in strata, the ideas in each stratum being linked by a complex of relations both among themselves and with those above and below. The lower the stratum, the deeper (and in general more difficult) the idea.”

Just as with the concept of ‘generality’, again, Euler’s sums should help us understand what Hardy is trying to explain when he talks of ‘depth’. Euler’s idea relates mathematical concepts that belong to different strata. In the entire procedure followed by Euler there is an underlying concept of infinity (a metaphysical stratum, we could say). There is also the stratum of the arithmetic, as the problem is proposed in terms of natural numbers, the inverses of the squares of which we want to sum. The calculation of this sum brings a geometric stratum into play, as the value of the sum is expressed by means of the square of the number π , which governs the geometry of the circle. Finally, the whole idea is based around the development of functions into infinite sums and products, one of the features of complex variables. And this rich tapestry of relationships between such diverse strata has been presented by the apparently simple Eulerian idea. This is what Hardy means when he talks about the depth of an idea – its ability to inescapably, luminously and fruitfully interlink various different mathematical strata.

Unexpected, inevitable, economic and illuminating

To the two intrinsic properties of generality and depth, Hardy added another three that can give a mathematical idea its aesthetic value. This time they are not properties but abilities – in other words, provisions that measure the idea’s ability to produce a particular aesthetic answer. Hardy called them “unexpectedness”, “inevitability” and “economy”. He described these qualities in the following way: “The arguments take so odd and surprising a form (quality of unexpectedness); the weapons used seem so childishly simple when compared with the far-reaching results (economy); but there is no escape from the conclusions (quality of inevitability).”

All these quality are easily detected in Euler’s sums.

On the one hand, the simplicity itself of the Eulerian idea undoubtedly makes it unusual, which lends Euler’s operations quite an ability to surprise (how something

so simple can lead to such profound results). Also, I am sure you will agree about the completely surprising nature of the result to which Euler's calculations lead. Before they reveal the solution to us, it is of course difficult to imagine that adding the even powers of the natural numbers we would come across the number π . All this is precisely what Hardy meant when he talked of the unexpectedness of a mathematical idea.

And the inevitability of their conclusions is no less recognisable in Euler's ideas. In view of Euler's simple and impeccable idea, the value of $\pi^2/6$ for the sum of the inverses of the squares of the natural numbers is completely inevitable.

And finally, the economy of Euler's procedure is also clear – in just a few lines Euler manages to solve a profound and difficult problem which Leibniz, the Bernoullis (and maybe even Newton himself) could not. It is undoubtedly a perfect illustration of what the philosopher George Santayana described in his book *The Sense of Beauty* as “the expression of economy”. Our approval of the economy of things is elevated to the point of becoming aesthetic appreciation.

In a way, Hardy's three qualities are related to what Santayana called “wit” in *The Sense of Beauty*, or to what the mathematician Gian Carlo Rota called “an idea's capacity to enlighten”. (*Enlightenment* was the term used by Rota in the chapter of his book *Indiscrete Thoughts* titled *The Phenomenology of Mathematical Beauty*.) On the one hand, Santayana directly relates genius with depth: “It is characteristic of wit to penetrate into hidden depths of things, to pick out there some telling circumstance or relation, by noting that the whole object appears in a new and clearer light.” And on the other, according to Rota, an enlightening idea throws light onto concepts or problems to which it relates or which it solves, it clarifies them, allows us to understand them better. And that is precisely what happens with the idea brought into play by Euler of adding the inverses of the even powers of the natural numbers.

This capacity of mathematical ideas to enlighten has been prized by scientists, engineers and architects throughout history. And as an example I have here what the architect Le Corbusier wrote when he got stuck during a project: “The lack of a rule, a law, jumped out at me. That filled me with terror, as it made me see that I was working in complete chaos. It was then that I discovered the need to bring in mathematics, the need for a regulator. An obsession which, from that time, always occupied a small corner of my brain.”

Euler's infinities and Kant's sublime

I am going to dedicate the last two sections of this chapter to the *Introductio in analysin infinitorum*, the book by Euler from which I have taken the examples that have helped me to illustrate Hardy's reflections on the beauty of mathematics.

In the *Introductio* there are no differential or integral calculations. What Euler does in this book is, as advertised by its title, to show the readers how to handle infinitely large and small quantities. In particular, the basic functions are studied using infinite processes; series are developed, some into infinite products (for the first time in the history of mathematics) and all of this is used to various ends. Some are analytical – adding infinite sums, for example (such as the examples we looked at in the third section of this chapter) – others are related to the theory of numbers.

The metaphysics of infinity together with Euler's ability to demonstrate make the *Introductio* one of the most beautiful texts in the history of mathematics. To the point where, as I will comment later, it influenced one of the essential texts on aesthetics – *Critique of Judgement* by German philosopher Immanuel Kant, or, more specifically, the aesthetic character of that which is sublime, just as Kant stated in his book.

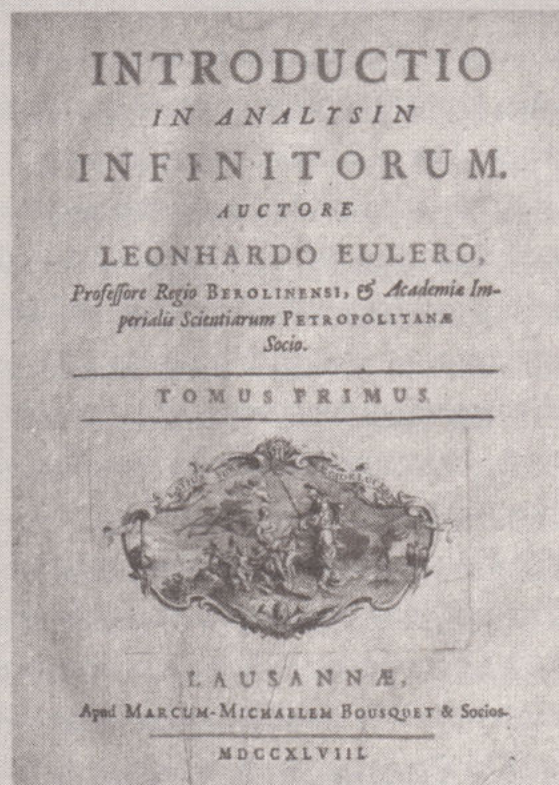
To bring you up to speed on the situation, I should start by briefly talking about the infinities as Euler understood them, and which give his book its title. So, in the *Introductio* Euler did not provide any definition of those infinitely small and large quantities (quantities that transcended all concepts of analysis during the 17th and 18th centuries, and a good part of the 19th), instead he dealt with them intuitively. Euler wanted those who read his book to develop certain skills for handling the infinities, a certain intuition on the way they work.

A brief description of the personality of these infinities, just as reflected in Euler's book, could be as follows. An infinitesimal quantity is an indefinitely small quantity that does not reach zero. The fact that it is non-zero allows it to appear on the denominator of coefficients, and since it is infinitely small, it can be taken to be zero when we wish to simplify expressions. On the other hand, an infinitely large quantity is invariant when added to a normal number, meaning that if N is one such infinitely large number, we have the following disconcerting inequality $N + 1 = N$. Equally, an infinitely small number w is a number which is not zero, but as many times as we repeat it, it will never exceed 1, $1/2$ or any other positive quantity we may imagine. To reach 1 with an infinitely small number w an infinitely large number N needs to be introduced such that $N \cdot w = 1$.

INTRODUCTIO: ONE OF THE THREE CLASSIC MATHEMATICAL TEXTS

The *Introductio* is not just any book – it is a fundamental text in the birth of mathematical analysis. An article in 1969, by mathematics historian Carl Boyer, which reviewed history's most important mathematical texts, included *Introductio* in a triad formed by none other than Euclid's *Elements* and Al-Khwarizmi's *Algebra*: "The most influential mathematics textbook of ancient times is easily named, for *Elements* by Euclid has set the pattern in elementary geometry ever since. The most effective textbook of the mediaeval age is less easily designated; but a good case can be made out for the *Algebra* of Al-Khwarizmi, from which algebra arose and took its name. Is it possible to indicate a modern textbook of comparable influence and prestige? Some would mention *Géométrie* by Descartes or *Principia* by Newton or the *Disquisitiones* of Gauss; but in pedagogical significance these classics fell short of a work by Euler titled

Introductio in analysin infinitorum. Here, in effect, Euler accomplished for analysis what Euclid and Al-Khwarizmi had done for synthetic geometry and elementary algebra respectively. Coordinate geometry, the function concept, and the calculus had arisen by the 17th century. Yet it was the *Introductio* that in 1748 fashioned these into the third member of the triumvirate – comprising geometry, algebra, and analysis."



The cover of the first edition of the *Introductio in analysin infinitorum* by Euler, published in 1748.

"Hardly any other work in the history of mathematical science gives to the reader so strong an impression of the genius of the author as the *Introductio*", wrote E.W. Hobson on *Introductio*. And, possibly, anyone who has read Euler's book would completely agree with what Hobson says. So, this impression of Euler's genius arises because the *Introductio* has enormous emotive capacity. It is, without doubt, a text

of great impact, because the genius shown by Euler in the book translates into a text loaded with beauty, and so it has a permanent impact on anyone who reads it. As we have seen, the way in which Euler handles the infinities in the *Introductio* is purely intuitive. This is precisely where the genius referred to in the Hobson quote resides. Infinities are dangerous entities with strange properties; their handling, if not done with due care, can have disastrous consequences. For the Greeks, infinity was a kind of fearsome beast, let's say a gigantic Minotaur that was to be avoided. But Euler did not flee; quite the opposite, he approached the beast, patted it on the back and yoked it, which allowed him to make a previously sterile field fertile. The docility shown by infinity in the hands of Euler is admirable. A docility, given the idiosyncrasy of the concept, almost shocking or chilling. And right there, in that shock and chill is the aesthetic achievement. I remind you of what Voltaire said: "It is not sufficient to see and to know the beauty of a work. We must feel and be affected by it." The German philosopher Theodore Adorno also stated that the aesthetic achievement of an object specifically resides in its capacity to shock, to



Portrait of Immanuel Kant (1724–1804), one of the most important philosophers in the history of humanity.

produce a kind of chill. And this idea appears in the famous conferences given by Serge Lang to the public at the Palais de La Découverte in Paris in the early 1980s under the title *The Beauty of Doing Mathematics*. There Lang referred to the “chills in the spine” produced by the most beautiful mathematical reasoning.

In his work on aesthetics, *Critique of Judgement*, the philosopher Kant introduced the category of aesthetics of the sublime. Kant (1724–1804) was of the generation after Euler, and he was born and lived practically all his life in Prussian Königsberg (which is now Kaliningrad, Russia). Königsberg is one of the Eulerian cities, together with Basel, where he was born, and St. Petersburg and Berlin, where he worked as a mathematician in the respective local academies of science. Euler did not live in Königsberg, but he did resolve the famous problem of its seven bridges. In the 17th century there were seven bridges in the city which linked the banks with two islands on the River Pregel, and people wondered whether it was possible to travel over all of them without crossing the same bridge twice. Euler, with a piece of reasoning as simple as it was imaginative, and which later gave rise to graph theory, showed that this route was impossible.

Taking into account Kant’s description of the sublime, it would not be exaggerating to say that the inspiration for defining that aesthetic concept could well have been found by Kant in Euler’s reasoning on infinitesimal quantities (those of Euler, or of any other 18th-century mathematician; the difference is that Euler was the one, by some margin, who best demonstrated the power of infinity). In *Critique of Judgement* Kant wrote: “The Sublime is that which in comparison with everything else is small; it is that which the mere ability to think shows a faculty of the mind surpassing every standard of Sense. The feeling of the Sublime is therefore a feeling of pain [...], at the same time, a pleasure thus excited, [...] a vibration, [...] a quickly alternating attraction towards, and repulsion from, the same Object.”

Now, the phrases “which in comparison with everything else is small”, or that “surpassing every standard of Sense” that Kant uses in his definition of the sublime, are none other than a transcription of the unnerving formula $N + 1 = N$, which describes the property of an infinitesimally large quantity, and which Euler uses over and over again in *Introductio*. And that “feeling of pain” is none other than that produced in our mathematician’s heart when reading $N + 1 = N$, or seeing a quantity in the denominator of a division that disappears into zero two lines later. But, on the other hand, the Kantian “pleasure thus excited” describes the perfection that one feels at the wonders achieved by Euler with those tormenting properties of the infinities. All of which, while reading Euler’s operations in the *Introductio*,

make us constantly feel “a vibration, a quickly alternating attraction towards, and repulsion from, the same Object”, those magical objects which dominate his book – the infinities.

With all the charm of geographical exploration

Euler’s reasoning was famed for not seeming particularly solid in terms of logical rigour. This is why, during the 19th century, mathematicians decided to substitute the infinitely small and large quantities with limits. The mathematics that we can read in *Introductio* do not appear to be especially rigorous. But appearances can be deceptive and today we know that analysis using infinitesimal quantities is just as rigorous as that which we now do using limits. Specifically, the logical grounding of the 18th-century analysis was carried out by Abraham Robinson in 1966 using model theory to construct a non-standard extension to the first-order theory of real numbers. This is now known as non-standard analysis.

Anyway, I do not think Euler would have lost much sleep over the lack of rigour in his mathematics, because Euler, like Descartes, Newton or Leibniz before him, was more concerned with discovering than demonstrating. He makes this abundantly clear in the prologue to the *Introductio*, in which there are continuous references to the act of clarifying, resolving, solving and inventing. So many references to the act of discovery are in contrast to the lack of any mention of demonstrating or proving.



The mathematician Abraham Robinson (1918–1974), who developed non-standard analysis.

And Euler's *Introductio* is written in such a way that the mathematics it contains unfolds before us as the wonders of nature unfolded before the spellbound eyes of explorers – and a world away from those extremely boring, tatty and forced pieces of logical-deductive reasoning that are so abundant in today's textbooks. Reading the *Introductio* is like stepping into a geographical exploration of the unknown. It reminds me of Antonio Pigaffeta's historical notes on Magellan's trip round the world, or those of Juan Sebastián de Elcano when he took over the expedition. In the *Introductio*, Euler does not deny us unsuccessful, but illustrative, attempts to resolve certain problems, just as Pigaffeta's narration does not hold back on Magellan's many failed attempts to get from the Atlantic to the Pacific before discovering the correct route.

There is no doubt that the *Introductio* is an initiatory journey to the world of the infinities, and that with his text Euler manages to make us feel the same dizziness for adventure transmitted to us by reading those notes on the first trip round the world. One more reason to read the *Introductio*: it is possibly the best historical work for discovering what creative genius means in mathematics, and one of the most suitable for feeling the commotion generated by its strange beauty.

Chapter 5

History and Beauty

At the end of the introduction to his famous *The Story of Art*, Ernest Gombrich upheld the importance of discovering the history of art because that knowledge helps to comprehend why artists proceeded as they did – or set about producing particular effects. A knowledge of the history of art, Gombrich told us, increases our sensitivity to capturing the finest nuances of difference and allows the aesthetic appreciation of art. To put it another way, the appropriate cultural baggage of a particular work of art allows the appreciation of its beauty, and the knowledge of history is an integral part of culture. The purpose of this chapter will be to apply this argument to mathematics. Its approach, therefore, falls into the category of contextualism. This aesthetic position states that a work of art should be considered in its context – historical, social, religious, cultural, etc – the more complete the context, the better. Therefore, sufficient historical knowledge enhances the context. The opposite approach, isolationism, ensures that a work of art must be self sufficient and that the more the knowledge of its context is needed in order to appreciate it, the more deficient it will be (so any isolationists should stop reading here).

From *Venus of Willendorf* to Duchamp's readymades

This final chapter revisits the thread running through the first two, so before going on to discuss mathematics I am going to make an illustrative incursion into the territory of the fine arts. In this case I am going to give a quick demonstration of how the history of art can help us to appreciate the beauty of a particular sculpture.

I will begin the journey with *Venus of Willendorf*. It is undeniable that a certain part of the beauty and fascination with this sculpture lies in the fact that we know it is ancient – its age is estimated at around 25,000 years. We know, because we know the history of art, that it is one of the oldest sculptures that has survived and this gives it special value. It is debatable whether or not this added value is aesthetic value; what is indisputable is that it is historical value, and that knowledge of it aids in the aesthetic appreciation of the Venus.

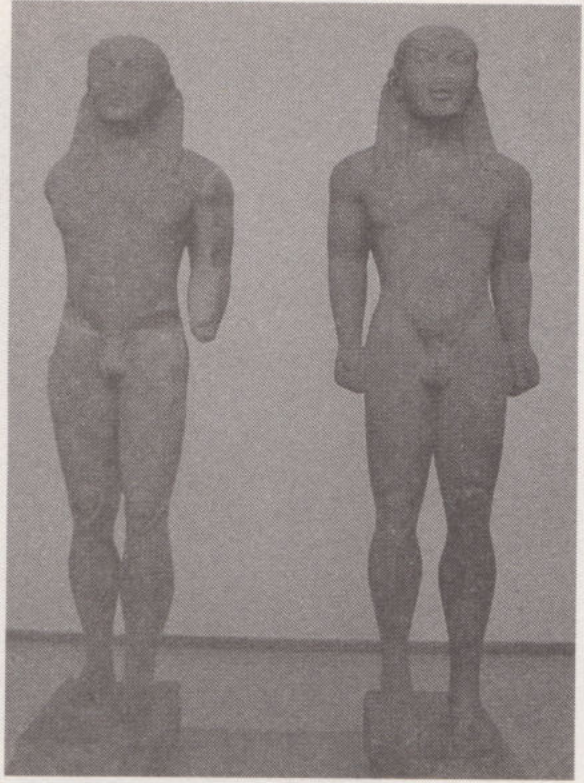
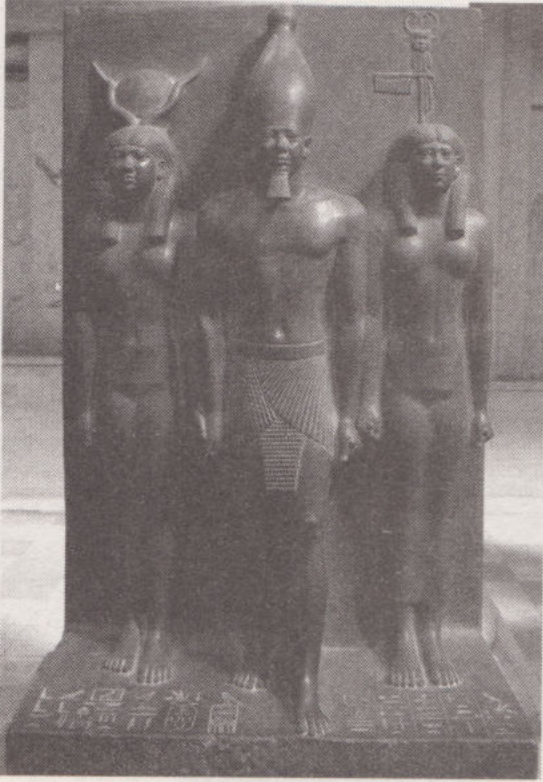


The Venus of Willendorf was discovered in 1908 by the archaeologist Josef Szombathy in the village of Willendorf, Austria. It is housed at the Naturhistorisches Museum in Vienna (photo: Matthias Kabel).

Equally, the history of art is responsible for informing us of the intentions, purpose and techniques of the sculptor, and what the sculpture's meaning is. In summary, history provides cultural enrichment that facilitates the aesthetic appreciation of the work.

For the correct consideration of Greek sculpture, is it necessary to know of its heritage – what the Greeks learnt, for example, from the Egyptians. It goes without saying that this knowledge is a contribution of history, and that its acquisition allows us to value the Greek offerings more and better appreciate the harmony and perfection they achieved in the representation of the human body.

It will also be history of art that is responsible for informing us of what might have happened during the Middle Ages for human representation to undergo the transformation represented in the following illustration. This juvenilisation of Romanesque sculpture with respect to the classic Greco-Roman sculpture; this apparent step back in the quality of the representation, in the perfection of the finish. Because the knowledge given to us by the history of art allows us a better appreciation of all Romanesque sculpture, by enlightening us on the new religious dimension that impregnated the representation of the human body. The all-powerful Catholic religion, omnipresent in mediaeval European society,



*A remnant of what the Greeks learnt from Egypt about sculpture:
on the left, the Egyptian sculpture set known as the Mycerinus Triad;
on the right, Greek sculptures dedicated to Kleobis and Biton.*

forced the subordination of everything human to God. Consequently the symbolic representation of this submission to divinity substituted the fundamental role of the naturalistic representation of the body in the Classical era.



*Detail of one of the sculptures in the Romanesque cloister of Millstatt Abbey,
in Austria, built in the 10th century.*

Again, it is the history of art that is responsible for explaining the reasons why sculpture once again recovered its classical canon, as the shape of man and woman returned to the artist's centre of interest. Its help will also be important when it comes to revealing the differences between classical sculpture and its evolution through to the romanesque Neoclassicism.



David by Michaelangelo (1501–1504) (photo: Rico Heil) and Venus Italica by Antonio Canova (1804–1812).

And, finally, in order to better appreciate the beauty of the new forms that began to appear in sculpture after the return of Classical realism, it is helpful to know the new goals set out by the artist. History is responsible for informing us about the rejection of the cold perfection of the finish of sculpture in favour of creating sculptures which were likely to have a greater impact on the spectator. How can we understand the process of acceleration undergone by art in the last century and a half without knowing the history of art? Is it possible to understand and appreciate Constantin Brancusi's version of *The Kiss* by his master Rodin, without any historical understanding of its palaeolithic inspiration?



The Kiss by Auguste Rodin (1889) and one of a series called The Kiss by Constantin Brancusi, which he began in 1908.

Is it possible to appreciate the beauty of the 20th century's artistic ideas (Duchamp's famous fountain, for example) without knowing the aesthetic value inherent the transgression.



Fountain (1917), Marcel Duchamp's most famous readymade. As you can see in the photo, the name which is signed on the piece is not that of Duchamp. The artist, consistently obtuse, signed as R. Mutt, the name of a German toilet manufacturer.

DUCHAMP AND THE READYMADES

Marcel Duchamp with his readymades (pre-existing objects) proposed that anything could be made into art if the artist so decided – a completely revolutionary notion aimed at the heart of the institution called art. Duchamp, who was certainly interested in mathematics and physics and, above all, chess, dedicated a lot of time to considering a question closely related to the aesthetic value of mathematics. Can one produce mental works of art not reliant on primarily retinal effects?

*Marcel Duchamp as Rose Sélavy, 1921.
Photo by Man Ray.*



From the Babylonians to set theory

The history of mathematics can help us appreciate the aesthetic value of mathematical reasoning in a similar way to the history of art helps us appreciate sculpture. Taking into account the greater difficulty of capturing the beauty of mathematics (as analysed in Chapter 2), the help of history is more necessary in the exact sciences than in any other artistic discipline.

Let's consider, for example, the following result: all triangles inscribed in a semicircle are right-angled. Diogenes Laertius attributes this theorem second hand to Thales, who sacrificed an ox when he discovered and demonstrated it, although Apollodorus, perhaps with more historical grounds, attributes it to Pythagoras.

It is an apparently straightforward result that can be demonstrated simply in several ways. It is, however, the history of mathematics that gives us the real key to the theorem. It was one of the first results that was accompanied by reasoning that sought to guarantee its general validity. In other words it was strengthened by a demonstration. A demonstration is nothing other than a logical dissection of the result until reducing it into universal and evident truths. In order to understand the magnitude of this fact (imposing the necessary company of a demonstration on the results) it has to be compared with the type of mathematical reasoning that was done by the Egyptians and the Mesopotamians, that is, to turn again to the history of

mathematics. History, as we go back in time, will transform the supposed simplicity of the theorem into a subtle one. We will even feel the conceptual proximity between the Greeks' procedures and the primitivism we see in Egyptian and Babylonian mathematicians. (Hardy said he once heard Littlewood say, "Greek mathematicians were not scholarship candidates, but fellows of another college.") Just like the *Venus of Willendorf*, this theorem has historic value, and the knowledge of that value is of assistance when it comes to its aesthetic appreciation.

There is another reason for the consideration of the theorem's history, which is related to the explanation in Chapter 3 about mathematics and its emotional circumstances. I am referring to the matter of the sacrificial ox. Although Diogenes Laertius attributes the sacrifice to Thales, most classical historians affirm that it was Pythagoras who offered a hecatomb when he discovered his famous theorem. The second definition of 'hecatomb' in the dictionary is "A sacrifice to the ancient Greek and Roman gods consisting originally of 100 oxen or cattle." The hecatomb attributed to Pythagoras could best be described as modest and not many oxen were sacrificed (although sources, among whom we can find Vitruvius, Cicero, Plutarch, Diogenes Laertius and several other authors, are not in agreement over this). If anyone thinks that this is not very many animals to sacrifice for the greater glory of these mathematical discoveries, they should also take into account the fact that a hecatomb was not held for just any reason. Furthermore, its causes were laden with hidden symbolic implications, although this is not why these reasons lost their connection to the more vital concerns of the human being, to the bottomless fears that have always clouded our minds, or our most intimate worries, dreams and miseries. Let's not forget that the hecatombs were a kind of basic sacrifice impregnated with religion, magic and mystery (if you will excuse the repetition), and they were offered in the hope of avoiding disasters and divine curses, in order to try and to a war, or to try to end an epidemic or a famine. The fact that hecatombs and everything that went with them, not just the ritual but also the blood and guts, appear as part of the geometric theorem should make you think. There will be those who argue that these hecatombs attributed to Pythagoras or Thales have little historical evidence and that they possibly never happened; that they are, in short, a myth. I would respond that, in that case, there would be even more to think about – why did Vitruvius, Cicero, Plutarch, Diogenes Laertius and so many others, all professional and undoubtedly busy men, take on the work of designing or transmitting a myth, and one which was so unusual, in order to praise something that was so apparently insignificant as a theorem? Why bother associating the intellectual

birth that gave rise to Greek mathematics with such a carnal and bloody event? Why link a purely abstract process with something as emotional as a ritual culling?

Just as with Greek sculpture, it will be history that allows us to better evaluate, and aesthetically value, the evolution of Greek mathematics from the simplicity of the first theorems to the depth and sophistication it would later achieve. And a good measure of the difference can be obtained by comparing the above-mentioned theorem with Archimedes' calculation of the area of the segment of a parabola described in Chapter 1.

In order to consider the aesthetic value of something that is apparently more ugly than Greek synthetic geometry, such as positional and zero notation, or the basic procedures of algebra, it would help, just as with Romanesque sculpture, to know that it lies in the apparent simplicity. Specifically there is a lack of sophistication that hides its aesthetic value. And it is here that history comes in to tell us that such simplicity, or modesty, is no more than a question of education. Geometry and algebra seem simple to us because they have become part of our school education. How could we not judge our own numbering system to be sophisticated when comparing it with that of the Greeks, or, on the other hand, how else would we explain that they had almost completely no algebraic techniques? How can we judge the conceptual difficulty of the numbering system, or that of algebra, without knowing its slow and arduous historic gestation process? Would it not also be essential for their correct evaluation to know the importance they both had in the 17th-century birth of much more sophisticated theorems, such as those of analytical geometry and, later, differential and integral calculus.

The same aesthetic evaluation of infinitesimal calculus requires knowledge of its origins – how, it was conceived by a series of advances on Greek mathematics and how it contributed to the consolidation of the scientific revolution. It could be said that it set science on the path to success that it enjoys today, sparking the development not just of mathematics but also of physics. The history of mathematics allows us to understand and evaluate all of this.

Gombrich wrote in his *The Story of Art* that both modern and ancient art came about as the artists' response to various specific problems. Thus, the revolutionary and innovative processes that so radically changed art in the mid-19th century came about when painters, above all, wondered why they had to content themselves with painting (as faithfully as they knew how) what they had in front of them, be it a natural landscape or a group of humans. Then came the dilemma about what the artist's role was. Was it simply to be a silent witness whose function is to faithfully reflect what the

eye sees (as if the painter was a mere camera), or was it to become the protagonist by reflecting internal emotional experience generated by what they were seeing? Using the creative freedom of the artist as one of its main arguments, art chose the second option and constructed a new world, criticised at times, disregarded at others, and mostly misunderstood, through which paraded impressionists, expressionists, abstract art, avant-garde artists and experimental artists. In this new and complex scene, which was often strange to the point of being outlandish, changing like never before, the history of art becomes a tool that is almost as essential as the sense of sight is for aesthetic evaluation and appreciation.

At the end of the 19th century mathematics also began a process of abstraction, which culminated in the insertion of nearly the whole of mathematics into the apparently aseptic and neutral field of set theory. The great driver of this process was the German mathematician Georg Cantor, and the motor that powered it was his passion for understanding and controlling the most feared monster of the mathematical bestiary – infinity. As if wanting to demonstrate that certain patterns of uniformity appear even in the most disconnected aspects of culture, or to justify Marcel Proust's phrase that everything from the same period is alike, the evolution of mathematics has very suggestive and illuminating similarities with what happened in the same decades with painting, sculpture, architecture, music and literature. It is with good reason that Cantor's saying, "The essence of mathematics is in its freedom," is parallel to the motif that was tirelessly repeated by painters, sculptors and composers to justify a good part of the artistic ideas.

I will dedicate the final pages of this book to infinity, its master Cantor – and of course his story.

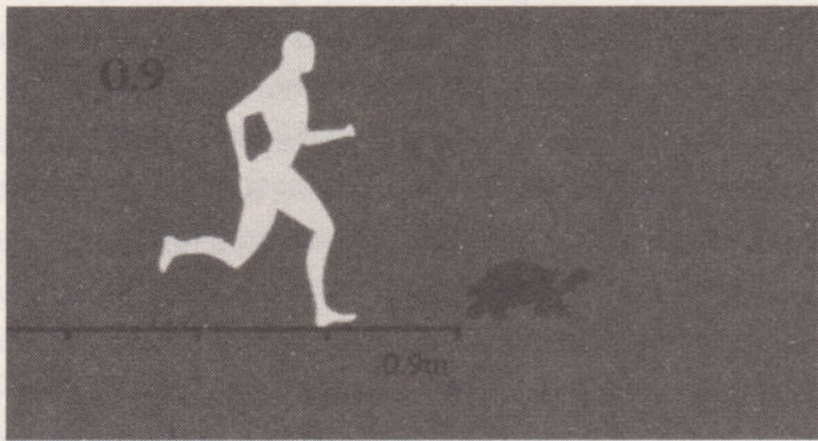
The viper in its nest

Infinity is like a nest of vipers, and it has taken human intellect several millennia and many attempts to be able to put a hand in it. Infinity is a product of the logical structure of our brain. This structure has the ability to create concepts through the denial of other existing ones. It starts from what our senses appreciate, the concept of that which has an end, termination, limit – in other words, of the finite, of the limited. We arrive at the concept of infinity by the denial of the finite and not through experience of the infinite. The paradox is that, despite being a product of the logical structure of our brain, infinity shows a certain incompatibility with logic, or rather, with common sense – which is often wrongly confused with logic.

Common sense is conditioned by the information provided to us by our senses, and this is necessarily information on, or from, the limited. Thus the infinite has very little respect for common sense, and we will get bitten by the monster every time we attempt to apply the former to the latter.

Therefore, it is not surprising that infinity instilled such fear in the Greeks, who made logic and common sense the key to their cultural revolution.

The story of Achilles and a tortoise, Zeno of Elea's puzzle of plurality and movement, is as much part of the collective imagination as Pythagoras' theorem, Don Quixote and Beethoven's Ninth Symphony. In order to build his paradox, Zeno uses infinity covertly – the same infinity that oozed through the Pythagorean crisis created by the discovery of the irrationals. The fact that $\sqrt{2}$ was not a fraction, and that infinities were needed in order to discover the exact value of the number had the unmistakable stench of the monster.



The story of Achilles and the tortoise represents the Greek philosopher Zeno of Elea's approach to the concept of infinity. The illustration shows a frame from Takeshi Kitano's film which is named after the famous paradox.

But infinity was a product of logic, of its capacity to generate concepts through denial. And it also had a friendly face. There were mathematical objects that manifestly paired badly with the finite. These include those numbers that are so prolific that nobody would deny that they are endless; or straight lines, which can be mentally stretched and lengthened without any apparent limits.

Given the circumstances of the infinite, in which the absurd, and from there the scandals and controversies, mixed with a certain undeniable yet tenuous simplicity, the Greeks decided to separate infinity into two parts – one was the acceptable 'potential infinity'; the other was the abhorrent 'actual infinity'. The Aristotelian 'common sense' was the knife that was used to cut into the monster. As it was

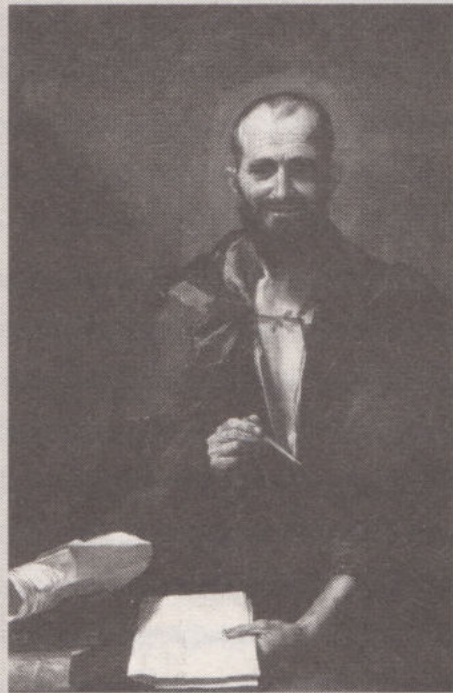
impossible to deny the existence of endless processes (the successive division of a segment at its middle point, or the never-ending increase of numbers), in Book III of *Physics*, Aristotle gave potential infinity the seal of approval: "To suppose that the infinite does not exist in any way leads obviously to many impossible consequences," such that "the infinite has a potential existence." In other words: "A thing is infinite either by addition or by division." However, infinity itself, as a complete thing, would not be permitted: "The infinite cannot be an actual thing and a substance and principle. [...] Magnitude is not actually infinite. But by division it is infinite." Aristotle's regulation impedes considering the numbers that form a whole to be finished, although it admits that numbers can be as large as we wish. It prohibits considering a segment as a collection of infinite points in a line, but it does allow dividing the segment in half, let's say, as many times as we want.

But the use of actual infinity also defied Euclid, who in his *Elements* included as a common notation or axiom that "The whole is greater than the part." This axiom is created by our intuition of the limited. Galileo, in his *Discorsi* (1638), was best able to explain this incompatibility of infinity with the Euclidean axiom. As each number generates a different square (2 generates 4; 3, 9; 4, 16, etc.) and in turn each square is generated by a unique number, Galileo argued, we can square the numbers

ARCHIMEDES AND INFINITY

Archimedes was perhaps the only Greek mathematician to break Aristotle's prohibition of actual infinity; although he did so measuredly and with self control. In Archimedes' reasoning for the calculation of the area of a segment of a parabola explained in Chapter 1, actual infinity comes into play. Whenever a surface is considered to be formed by an infinite collection of straight segments, "The straight lines drawn in a triangle constitutes the triangle itself". However, Archimedes was not seeking to spark a revolution against Aristotle because, as he himself stated, his arguments "were far from proof".

Archimedes according to the vision of painter José de Ribera (1591–1652).

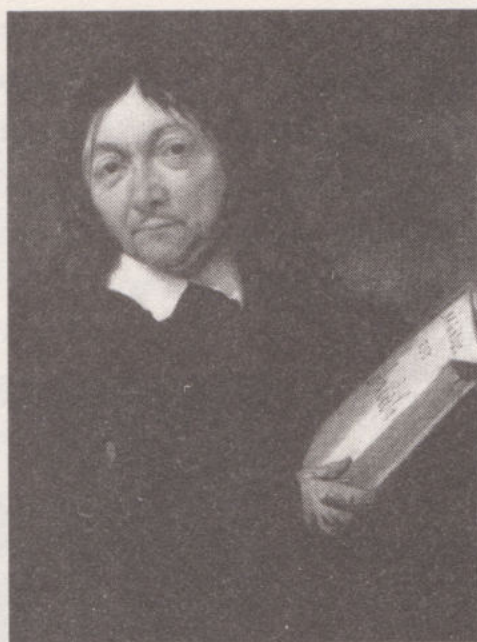


with their squares concluding that there will be as many squares as numbers; but it is patently obvious that the square numbers are only a part of the numbers (2, 3, 5, 7 are numbers, but not squares) and, consequently, we conclude that there are more numbers than squares (the whole is greater than the part). And, so, we find ourselves with the paradox that the numbers are, at the same time, both as many as and more than the squares. Galileo concludes that “the attributes of greater, less and equal do not apply to the infinities”.

The association of the Christian God with infinity helped in overcoming the ancient Greek fear of the monster, and infinity, potential and actual, was definitively installed in the heart of 17th-century mathematicians. According to Aristotle, the Scholastics denied man an understanding of actual infinity, although they were left with no option but to introduce it into theological debates. Thus, Thomas Aquinas considered God as the first and total infinity – actual infinity. The concept appears quite frequently in the work of the philosophers of the 17th century, some of whom, we should not forget, were also scientists and mathematicians. Thus, in Descartes we find: “By ‘God’, I understand a substance that is infinite, independent, supremely intelligent, supremely powerful,” and also: “We reserve the qualification of infinite only for God, both because we see no limits to his perfections, and also because we are so sure that he cannot have them.” Meanwhile Spinoza said: “I understand by God the Absolutely Infinite Being; that is to say, substance constituted by an infinity of attributes, each of which expresses an eternal and infinite essence.” while Leibniz expressed it thus: “We may also hold that this supreme substance, which is unique, universal and necessary, must be illimitable and must contain as much reality as is possible.”

No prohibition can do away with what is truly useful. And no concept has fertilised mathematics as infinity did, as nothing has made mathematics as useful at explaining the phenomena of nature. Infinity is the fundamental ingredient of infinitesimal calculus, and infinitesimal calculus is, without doubt, the most powerful and efficient tool for the study of nature that mathematicians have ever developed – which is no more than another demonstration of the paradoxes that surround infinity. How is it possible that, despite being no more than a sub-product of the logical structure of our brain, infinity is so essential for scientific understanding of the Universe that surrounds us?

Despite this unquestionable, although incomprehensible, usefulness, the bad press received by actual infinity did not cease, and even the so-called Prince of Mathematics joined in. The great Gauss wrote: “I protest against the use of infinite



Everyone from the mediaeval Scholastics, such as St. Thomas Aquinas (ca. 1224–1274), to the rationalist philosophers, such as Descartes (1596–1650), had to confront the conceptual problem of ‘actual’ infinity.

magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit that certain ratios approach indefinitely close, while others are permitted to increase without restriction.”

Cantor and the libertarian essence of mathematics

Practically while Gauss was writing those words, Georg Cantor (1845–1918) was born. Cantor made compliance with infinity possible, due to the way in which the monster allowed itself to comply.

Cantor started by comparing the various infinite sets of numbers available to him. To compare infinities he used the technique for coupling elements. If you can couple the elements in a set with those of another, such that there is no excess in one nor lack in the other, then those two sets have the same number of elements.

Thus, he managed to pair the natural numbers with the integers and, also, with the fractions. Contrary to what logic tells us – the whole is greater than the part – Cantor’s operations showed that there are as many natural numbers as fractions.

But, in order to be able to do mathematics with infinity, Cantor needed a certain variety, he needed infinite sets of different sizes. He found the first decisive result at the end of 1873, when he found two infinities that he could not couple. Specifically,

Cantor showed that the set of natural numbers cannot be paired with those of the points on a segment. It is without doubt one of the most revolutionary results in the history of mathematics. For this result, as transcendent as it is profound, Cantor found an extremely simple and elegant demonstration. Just as with the paintings of the impressionists, its taste will be enhanced when dressed with sufficient knowledge of its history and circumstances.



Cantor in 1894, at 49 years of age, at which time in which he was trying to systematise his set theory.

Cantor's demonstration

To simplify, instead of the points on a segment, I am going to consider the infinite binary representations of the type $0.a_1a_2a_3\dots$, where each digit $a_1, a_2, a_3\dots$ of the representation is worth 0 or 1. It is not difficult to see, although it may be a little technical for this book, that there are as many points on a segment as binary representations of this type.

Cantor's proof uses a diagonal procedure for generating a binary representation, given any coupling between the numbers $1, 2, 3, 4\dots$ and the binary representations, which does not appear as the pair of any number. Imagine that we have any coupling of the numbers and binary representations. In order to simplify we take the following, which is enough to show the first pairs.

$$\begin{aligned}
 1 &\leftrightarrow 0.\boxed{1}0010011\dots \\
 2 &\leftrightarrow 0.1\boxed{1}011110\dots \\
 3 &\leftrightarrow 0.00\boxed{0}10001\dots \\
 4 &\leftrightarrow 0.111\boxed{1}1011\dots \\
 5 &\leftrightarrow 0.0101\boxed{0}100\dots \\
 &\vdots \qquad \qquad \vdots \\
 ? &\leftrightarrow 0.00101\dots
 \end{aligned}$$

Notice that I have put a square around certain digits: the first digit in the first representation, the second in the second representation, the third in the third representation, etc., and I have constructed a new representation (that which appears at the end of the list after the vertical ellipses) by modifying those digits: where there was a 0 I have put a 1 and where there was a 1 I have put a 0. (Thus, the first digit of the new representation is 0, the second is also 0, the third a 1, the fourth a 0, and so on.)

With this selection I have guaranteed that, whatever digits follow, the representation I am constructing is different from those that appeared before, as the new representation differs from the first in its first digit, from the second in its second digit, from the third in its third digit, and so on. This should convince you that the above list has no pair for the new binary representation. Also, you will soon see that Cantor's procedure does not depend on the previous list. If the list changes I can adjust the procedure to the new list without any trouble and generate a new representation without a pair.

Absolute infinity and Cantor's legacy

After dedicating a quarter of a century to his study, Cantor managed to arrange the universe of the infinities. As if it were a parade of monsters, he arranged them one after the other in a similar way (figuratively speaking) to how the numbers are ordered, and stipulated how they could be added, multiplied, some infinities raised to the power of others, etc. Cantor, however, did not completely master the infinities. There are still some incompletable multiplicities, an absolute infinity as Cantor called it, which is free from mathematical (let's not use the term logical any more) control or mastery: "The Absolute", wrote Cantor in 1883, "can only

CANTOR'S DIAGONALISATION PROCEDURE

This very procedure, together with the concept of parts of a set, allowed Cantor to show how to construct infinite sets of any size. You will get a clear idea with a simple example: imagine you have the set $A = \{1, 2, 3\}$ formed by the three numbers 1, 2, 3. The set of the parts of set A is the new set that is contained by considering all the sets which we can form with the elements of A , including empty set $\emptyset$; if we call it $P(A)$, then:

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

If we start with an infinite set A , Cantor proved that the infinity corresponding to the set of parts of A is always greater than the initial one. And he demonstrated it using his diagonal procedure again. The idea is the same, but it has to be adapted to the new situation. Let's take a coupling of a set A with the set of its parts $P(A)$. Each element x of A will have a set X as a pair formed by elements of A . We now define a part of A which will not have a pair: It is the set that I call Y and is formed by the elements x of A that are not in their corresponding pair X . Indeed, if an element x of A is in its pair X , then, by definition of Y , that x will not be in Y . Then $X \neq Y$, so x is an element of X but not of Y . On the other hand, if an element x of A is in its pair X , then by definition of Y , that x will be in Y . Again $X \neq Y$, so x is an element of Y but not of X . This shows that no element x of A can have the set Y as a pair.

be recognised, never known, not even approximately." The infinity that interested Cantor is located between the finite and the absolute.

Cantor emerged from his showdown with infinity reasonably victorious, but not without serious injuries. Because of his research into infinity, Cantor attracted the animosity of a large part of the German mathematical establishment. In 1879 he obtained a chair at the University of Halle, where he would stay for the rest of his life, despite his interest in getting to Berlin or Göttingen, better universities that closed their doors to him. Despite the occasional lack of tact on Cantor's part, there is little doubt about the cause of his harassment from influential colleagues. His research was often qualified as insignificant and lacking any interest, and when his efforts began to be recognised, rumours attributed the following slur to Henri Poincaré (1854–1912), one of the period's mathematical giants: "Future generations will consider Cantor's set theory an illness to recover from."

THE SET OF ALL THE SETS AND OTHER SIMILAR MONSTERS

Absolute infinity is related to those immense and unimaginable principles such as the set of all sets, or the set of the sets that are members of themselves, to give just a couple of examples. The latter is the subject of the paradox found by Bertrand Russell in 1901: does the set formed by those sets that do not belong to themselves belong to itself? If it belongs to itself, it contradicts the fact that the set is formed by the sets that do not belong to themselves, but if it does not belong to itself then, by definition, it would have to belong to itself.

But Cantor never concerned himself with these paradoxes because he was convinced that they did not affect the type of sets and infinities that he studied, but those enormous entities related to the absolute – that thing we can recognise but not know.

The paradoxes came about by the somewhat ingenuous consideration that a set is any collection of things. The formulation of this paradox is similar to those that can be used to contradict the all-powerful nature of God – another ingenuity. Can God create a stone that He is not capable of lifting? If He can create it then there is something which God cannot do: lift the stone; if He cannot create it, then there is also something that He cannot do.

Mathematics, however, ended up justifying Cantor. Little by little it started to recognise the importance of his work, to the point that the revolution it began changed the way we do mathematics. Cantor started a process of abstraction characterised by the pairing of proofs of non-constructive existence. In other words, after Cantor we mathematicians have admitted that the existence of a determined mathematical object can be guaranteed even when a way of constructing such an object is not specified. Cantor's efforts culminated in the creation of set theory, the territory where David Hilbert, the period's most influential mathematician, finally placed the paradise of mathematics.

The collapse of a genius

Georg Cantor was an exultant and excessive figure, and perhaps because of this he was also emotionally unstable. In mid-1884, then nearly 40 years of age, he suffered his first mental breakdown. All we know about this crisis is that it lasted for about two months and it went as suddenly as it came. From that time, Cantor's personality, already controversial and with pronounced mystical tendencies, became even more

eccentric. In a way he abandoned mathematical research as his main activity, and he dedicated his time to other, rather more doubtful, intellectual occupations. As a result of these studies he came to maintain and uphold that Francis Bacon was the author of the works of Shakespeare and that Joseph of Arimathea was the father of Jesus Christ. The nervous breakdowns recurred, above all after 1899, a consequence of which was that he was admitted to a rest home every two or three years with paranoia and symptoms of manic depression. It was not a surprise that he received the news of the prizes and honours awarded to him at the University of Halle asylum.

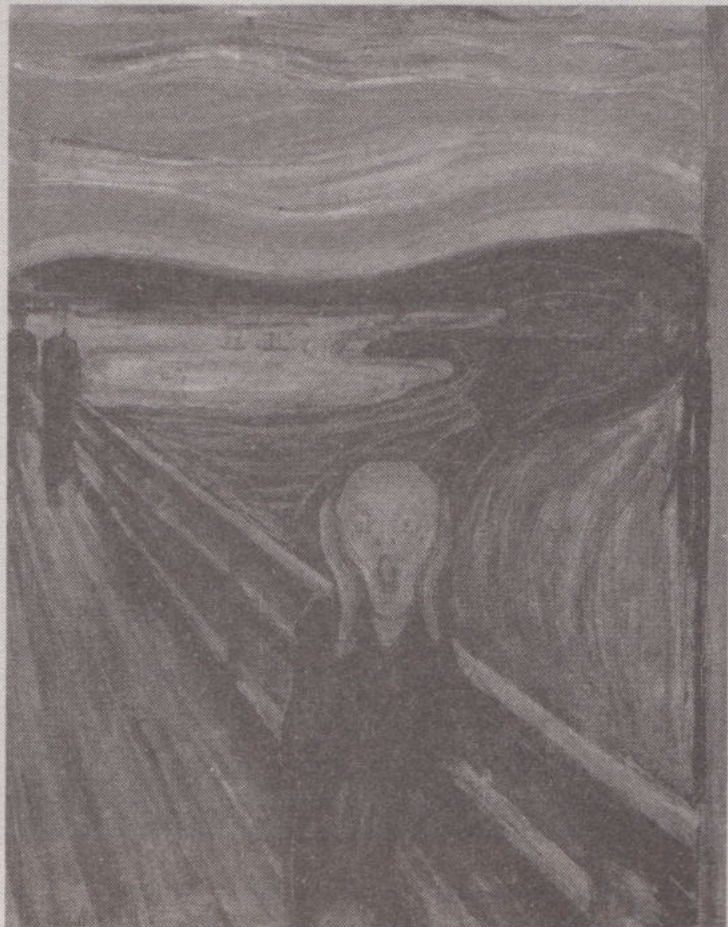
In the folklore of the history of mathematics those collapses are attributed to his battle with infinity. Was it his clash with the infinity monster that really caused Cantor's mental illness? It is undeniable that a certain melodramatic tension may incline us to answer this question in the affirmative. And, as I have said, there have been more than a few authors who have been swept up by it to the point that it has become mathematical folklore. Perhaps it was Bertrand Russell who contributed most to the cause (maybe inadvertently). Cantor visited Britain for the first time in September 1911, having been invited to the celebrations of the 500th anniversary of St. Andrews University in Scotland. He later travelled to London and wrote a couple of letters to Russell proposing a meeting that in the end never took place. The fact is that Russell included in his autobiography, published in 1967–1969, a reference to those two letters and the behaviour of a somewhat eccentric Cantor (perhaps a consequence of the excitation caused the first time he entered into the realm of Shakespeare and Bacon). “Georg Cantor was, in my opinion,” wrote Russell “one of the greatest intellects of the 19th century [...] After reading the following letter no one will be surprised to learn that he spent a large part of his life in a lunatic asylum.” Regarding this comment, British mathematics historian Ivor Grattan-Guinness, one of the first to raise his voice against the mathematical reasons for Cantor's madness, wrote in 1971: “Cantor's two letters have an undeniable erratic nature. In fact, the manuscripts are even more relevant. They show various habits that he had when the agitation overcame him. The handwriting is very flowery and the lines tend to rise up as they progress across the page; and not only do they continue into the margins (something that was typical of Cantor), but on the second page of the letter he even wrote from top to bottom over other lines which, as is normal, went from left to right. There is even a paragraph of the letter that is written on the back of the envelope.”

CANTOR AND MUNCH

"My friends went on walking, while I lagged behind, shivering with fear. Then I heard the enormous, infinite scream of nature." This is how the Norwegian painter Edvard Munch described the sensation that led him to paint *The Scream*, his most famous painting. And, indeed, the painting's symbolic power, the dramatic use of perspective, unreality and violence of the colours, also suggest aspects of excess, which are close to the infinite, to the infinite fear of the screamer. A scream that can be seen but not heard, and not hearing the scream, which can nevertheless be felt, produces fear because the threat of hearing it at any moment keeps us in suspense.

Munch's work caused an artistic dispute in Germany similar to the mathematical dispute generated by Cantor's work – and at a similar time. On 5 November 1892, Munch's exhibition in Berlin opened. It closed a week later giving rise to intense controversy, known as the 'Munch Affair'. During the controversy the limits of freedom of the artist were argued acrimoniously and at length. Cantor was on the receiving end of similar criticism, which led him to say, "The essence of mathematics is in its freedom." Just as with Cantor's work on mathematics, that of Munch had an enormous influence on painting, and not only among German expressionist groups, but also on Picasso (in *Guernica* no less). There is still one final unsettling similarity between Munch and Cantor. Munch also had nervous breakdowns, although perhaps not as acute and persistent as those suffered by Cantor.

The best known version of Munch's painting The Scream (1893), is in the National Gallery of Oslo, from where it was stolen in 1994 but recovered in 2006.





Another photograph of Cantor; this time from 1917, when the mathematician was 72 years old and was about to be admitted to (or was already at) the Halle asylum where he would die a few months later.

It is very possible that Cantor's mental illness had genetic causes; although this does not mean that the strain of his academic relationships or the tension of the struggle with infinity did not sharpen or hasten his mental crises. Other tragedies of life may also have had an influence, such as when, in 1899, Rudolf, the youngest of his six children died, at the age of 13.

In May 1917 Cantor was admitted again, against his will, to the University of Halle asylum. According to Grattan-Guinness: "The war led to a lack of food, and Cantor lost weight and was a victim of hunger, as well as tiredness and illness [...]. At New Year, Cantor sent his wife the last 40 pages of the previous year's calendar, implying that he had lived those days; but on 6 January he died suddenly and painlessly of a heart attack. He was buried in Halle near his son Rudolf."

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The Poetry of Numbers

Revealing beauty in maths

The dictionary states that mathematics is a deductive science that studies the properties of abstract entities, such as numbers and geometric shapes, and their relationships. However, this definition misses out on a fundamental fact: more often than not, mathematics is guided by poetic emotion and a sense of beauty. The purpose of this book is to illustrate, using examples from the history of the discipline, the extremely important aesthetic and emotional dimensions in the science of mathematics.